Auslander-Reiten triangles in Frobenius categories and applications

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International Conference on Representations of Algebras

ICRA 21, 2024

Shanghai Jiao Tong University
Shanghai - China

August 6, 2024

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joint with

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Road map

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   - Preliminaries
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2. Applications
   - Shapes of Auslander-Reiten triangles in the bounded derived categories
   - A relation between the strong global dimension, complexes of fixed size and derived category
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Notations

- $\mathcal{C}$ is a Krull-Schmidt category.
- Λ is a finite-dimensional $\Bbbk$-algebra over field $\Bbbk$.
- For simplicity, we assume that Λ is basic and $\Bbbk$ is algebraically closed.
- Denote by $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ the standard duality on Λ-mod.

Preliminaries

- A **pair** (i, d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{C}$ is said to be **exact** if (X, i) is a kernel of d, and (Z, d) is a cokernel of i.
- Let $\mathcal{E}$ be a class of exact pairs closed under isomorphisms.
- If (i, d) in $\mathcal{E}$, the morphisms i and d are called an $\mathcal{E}$-**inflation** and $\mathcal{E}$-**deflation**, respectively and the exact pairs in $\mathcal{E}$ are called $\mathcal{E}$-**conflations**.
- We call the class $\mathcal{E}$ an **exact structure** on $\mathcal{C}$ and $(\mathcal{C}, \mathcal{E})$ an **exact category** (in the sense of D. Quillen (1973) or equivalently in the sense of Gabriel and Roiter (1984)).

- The sequence $X \xrightarrow{i} Y \xrightarrow{d} Z \in \mathcal{C}$ is called $\mathcal{E}$**-sequence** if $(i, d) \in \mathcal{E}$.
- An object $P \in \mathcal{C}$ is called $\mathcal{E}$**-projective** if for any deflation $d : Y \to Z$ and morphism $f : P \to Z$ there is $g : P \to Y$ such that $f = dg$.
- We say that the exact category $(\mathcal{C}, \mathcal{E})$ has **enough** $\mathcal{E}$**-projectives** if for all $Z \in \mathcal{C}$ there is a deflation $d : P \to Z$ with P an $\mathcal{E}$-projective.
- An exact category $(\mathcal{C}, \mathcal{E})$ is called **Frobenius** category if $(\mathcal{C}, \mathcal{E})$ has enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives and if moreover the $\mathcal{E}$-projectives coincide with the $\mathcal{E}$-injectives.

Definition

An $\mathcal{E}$-sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called **Auslander-Reiten $\mathcal{E}$-sequence** or **almost split $\mathcal{E}$-sequence** if if f is a minimal left almost split morphism and g is minimal right almost split morphism.

- $\underline{\mathcal{C}}$ is the stable category of $\mathcal{C}$, i.e., the objects of $\underline{\mathcal{C}}$ are the same as those in $\mathcal{C}$, and two morphisms $h, h' : X \to Y$ in $\underline{\mathcal{C}}$ are identified provided that $h - h'$ factors through a $\mathcal{E}$-projective in $\mathcal{C}$.
- $\underline{\mathcal{C}}$ is a Krull-Schmidt category.
- $\underline{\mathcal{C}}$ is a triangulated category category (D. Happel, 1988).

A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\underline{\mathcal{C}}$ is called an **Auslander-Reiten triangle** if the following conditions are satisfied:

(AR1) X, Z are indecomposable.

(AR2) $w \neq 0$.

(AR3) If $f : W \to Z$ is not a retraction, then there is $f' : W \to Y$ such that $vf' = f$

Theorem (Y. Calderón-Henao, F. Gallego-Olaya and —)

If $\mathcal{C}$ has Auslander-Reiten $\mathcal{E}$-sequences, then $\underline{\mathcal{C}}$ has Auslander-Reiten triangles.

Main result

Theorem (Y. Calderón-Henao, F. Gallego-Olaya, and —)

Assume that $\mathcal{C}$ has Auslander-Reiten $\mathcal{E}$-sequences. If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is an Auslander-Reiten triangle in $\underline{\mathcal{C}}$ with X, Y, Z without direct $\mathcal{E}$-projective summands, then there is an Auslander-Reiten $\mathcal{E}$-sequence in $\mathcal{C}$, $X \longrightarrow Y \oplus P \longrightarrow Z'$ with P an $\mathcal{E}$-projective and his induced triangle is isomorphic to triangle given.

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Repetitive Algebras

Repetitive algebra $\widehat{\Lambda}$ of Λ as proposed by D. Hughes and J. Waschbüsch (1983).

- The underlying vector space of repetitive algebra Λ is given by

$$\widehat{\Lambda} = (\oplus_{i\in\mathbb{Z}}\Lambda) \oplus (\oplus_{i\in\mathbb{Z}}D\Lambda),$$

  $\widehat{a} = (a_i, \varphi_i)_{i\in\mathbb{Z}}$ with $a_i \in A$, $\varphi_i \in D\Lambda$ and almost all a_i, φ_i being zero.

- The multiplication is defined by

$$\widehat{a} \cdot \widehat{b} = (a_i, \varphi_i)_{i\in\mathbb{Z}} \cdot (b_i, \psi_i)_{i\in\mathbb{Z}} = (a_i b_i, a_{i+1}\psi_i + \varphi_i b_i)_{i\in\mathbb{Z}}.$$

$$
\begin{array}{lccccccccc}
\widehat{M}: & \cdots \rightsquigarrow & M_{i-1} & \overset{f_{i-1}}{\rightsquigarrow} & M_i & \overset{f_i}{\rightsquigarrow} & M_{i+1} & \rightsquigarrow \cdots \\
\Big\downarrow \widehat{h} & & \Big\downarrow h_{i-1} & & \Big\downarrow h_i & & \Big\downarrow h_{i+1} & \\
\widehat{M}': & \cdots \rightsquigarrow & M'_{i-1} & \overset{g_{i-1}}{\rightsquigarrow} & M'_i & \overset{g_i}{\rightsquigarrow} & M'_{i+1} & \rightsquigarrow \cdots .
\end{array}
$$

f_i are Λ-homomorphims $f_i : D\Lambda \otimes_\Lambda M_i \longrightarrow M_{i+1}$, such that $f_{i+1}(1 \otimes f_i) = 0$.

$$
\begin{array}{ccc}
D\Lambda \otimes_\Lambda M_i & \xrightarrow{f_i} & M_{i+1} \\
\Big\downarrow 1\otimes h_i & & \Big\downarrow h_{i+1} \\
DA \otimes_\Lambda M'_i & \xrightarrow{g_i} & M'_{i+1}.
\end{array}
$$

- We denoted by $\widehat{\Lambda}$-mod the category of finitely generated left modules over the repetitive algebra $\widehat{\Lambda}$.
- We denoted by $\widehat{\Lambda}$-$\underline{\text{mod}}$ the stable category of $\widehat{\Lambda}$-mod.
- The category $\widehat{\Lambda}$-mod has almost split sequences (1983 D. Hughes and J. Waschbüsch).
- The category $\widehat{\Lambda}$-mod is a Frobenius, and the category $\widehat{\Lambda} - \underline{mod}$ is triangulated (1988 D. Happel).

$$\widehat{M} \xrightarrow{\widehat{h}} \widehat{M}' \xrightarrow{\widehat{h}'} \widehat{M}'' \xrightarrow{\widehat{h}''} T\widehat{M} \qquad (1)$$

Corollary (Y. Calderon-Henao, —, and J.A. Veléz-Marulanda, 2023, [2])

Assume that (1) *is an Auslander-Reiten triangle in* $\widehat{\Lambda}$-$\underline{\text{mod}}$*, such that neither* $\widehat{M}$ *nor* $\widehat{M}'$ *nor* $\widehat{M}''$ *has projective direct summands in* $\widehat{\Lambda}$-mod*. Then there exists an Auslander-Reiten sequence in* $\widehat{\Lambda}$-mod

$$0 \to \widehat{M} \xrightarrow{(\widehat{h},\widehat{\alpha})^t} \widehat{M}' \oplus \widehat{P} \xrightarrow{(\widehat{h}'',\widehat{\beta})} \widehat{M}'' \to 0, \qquad (2)$$

such that $\widehat{P}$ *is a projective* $\widehat{\Lambda}$*-module and* (2) *induces the Auslander-Reiten triangle* (1)*. Moreover, if* $\widehat{P} \neq 0$*, then* $\widehat{P}$ *is indecomposable,* $\widehat{M} \cong \operatorname{rad} \widehat{P}$ *and* $\widehat{M}'' \cong \widehat{P}/\operatorname{soc} \widehat{P}$ *as* $\widehat{\Lambda}$*-modules.*

Shapes of Auslander-Reiten triangles in the bounded derived categories

$C^b(\text{proj}\Lambda)$ the category of the complexes over $\text{proj}\Lambda$.

$$
\begin{array}{cccccccccccc}
M: & & \cdots & \longrightarrow & M_{i-1} & \xrightarrow{f_{i-1}} & M_i & \xrightarrow{f_i} & M_{i+1} & \longrightarrow & \cdots \\
\downarrow h & & & & \downarrow h_{i-1} & & \downarrow h_i & & \downarrow h_{i+1} & & \\
M': & & \cdots & \longrightarrow & M'_{i-1} & \xrightarrow{g_{i-1}} & M'_i & \xrightarrow{g_i} & M'_{i+1} & \longrightarrow & \cdots .
\end{array}
$$

$\widehat{\Lambda}$-mod the category of finitely generated left modules over the repetitive algebra $\widehat{\Lambda}$

$$
\begin{array}{cccccccccccc}
\widehat{M}: & & \cdots & \rightsquigarrow & M_{i-1} & \overset{f_{i-1}}{\rightsquigarrow} & M_i & \overset{f_i}{\rightsquigarrow} & M_{i+1} & \rightsquigarrow & \cdots \\
\downarrow \widehat{h} & & & & \downarrow h_{i-1} & & \downarrow h_i & & \downarrow h_{i+1} & & \\
\widehat{M}': & & \cdots & \rightsquigarrow & M'_{i-1} & \overset{g_{i-1}}{\rightsquigarrow} & M'_i & \overset{g_i}{\rightsquigarrow} & M'_{i+1} & \rightsquigarrow & \cdots .
\end{array}
$$

Definition

An $\widehat{\Lambda}$*-homomorphism* $\widehat{h} = (h_i)_{i\in\mathbb{Z}} : \widehat{M} = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow \widehat{M}' = (M'_i, g_i)_{i\in\mathbb{Z}}$*:*

1. *is called* ***smonic*** *(resp.* ***sepic****) if all its components* h_i *are split monomorphisms (resp. split epimorphisms) and*
2. *is called* ***sirreducible*** *if there is exactly one index* ι_0 *such that* h_{ι_0} *is irreducible morphism and* h_i *is a split epimorphism for* $i < \iota_0$ *and a split monomorphism for* $i > \iota_0$*.*

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Shapes of the irreducible morphisms

Theorem (— and H. Merklen, 2009, [4])

Theorem (—, 2018, [3])

Let $\widehat{h} : \widehat{M} = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow \widehat{M}' = (M_i, g_i)_{i\in\mathbb{Z}}$ *be an irreducible homomorphism in* $\widehat{\Lambda}$*-mod. Then one of the following conditions holds:*

1. *h is a smonic morphism;*
2. *h is a sepic morphism;*
3. *h is a sirreducible morphism.*

Shapes of Auslander-Reiten Triangles

Theorem (E.R. Alvares, S.M. Fernandes, and —, 2020, [1])

Theorem (Y. Calderón-Henao, —, and JA. Vélez-Marulanda, 2023, [2])

If $\widehat{M} \xrightarrow{\widehat{h}} \widehat{M}' \xrightarrow{\widehat{h}'} \widehat{M}'' \xrightarrow{\widehat{h}''} T(\widehat{M})$ *is an Auslander-Reiten triangle in* $\widehat{\Lambda} - \underline{mod}$*, then*

1. *If* $\underline{\widehat{h}}$ *is smonic, then* $\underline{\widehat{h}'}$ *is sepic.*
2. *If* $\underline{\widehat{h}}$ *is sepic, then* $\underline{\widehat{h}'}$ *is sirreducible.*
3. *If* $\underline{\widehat{h}}$ *is sirreducible, then* $\underline{\widehat{h}'}$ *is smonic or sirreducible.*

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A relation between the strong global dimension, complexes of fize size and derived category

The **strong global dimension** of Λ is defined as

$$\text{s.gl.dim}(\Lambda) \quad := \quad \sup\left\{\ell(X) | X \in \mathrm{K}^b(\text{proj}\Lambda) \text{ indecomposable}\right\}.$$

$\eta = \text{s.gl.dim}(\Lambda)$ and assume that $1 \leq \eta < \infty$.

$C_n(\text{proj}\Lambda)$, whose objects are the complexes X with $X^i = 0$ if $i \in \{1, \dots, n\}$, i.e., X is of the form

$$X : \cdots \to 0 \to X^1 \to X^2 \to \cdots \to X^{n-1} \to X^n \to 0 \to \cdots .$$

- $C_n(\text{proj}\Lambda)$ has almost split sequences (R. Bautista, M.J. Souto-Salorio, and Rita Zuazua, 2005).
- The Auslander-Reiten quiver of $C_n(\text{proj}\Lambda)$ can be constructed by using the well-known knitting algorithm which is also used in order to build the Auslander-Reiten quiver of $\text{mod}\Lambda$ (C. Chaio, I. Pratti, and M.J. Souto-Salorio, 2017).
- $C_n(\text{proj}\Lambda)$ is not a Frobenius category.

Definition

Let $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ be an indecomposable complex in $C_n(\mathrm{proj}\Lambda)$.

1. We say that X can be **extended to the left** in $C_{n+1}(\mathrm{proj}\Lambda)$ if there is a projective Λ-module ${X'}^0$ and a non-zero morphism ${d'}^0 \colon {X'}^0 \to X^1$ such that $d_X^1 {d'}^0 = 0$. We denote the extended complex to the left of X in $C_{n+1}(\mathrm{proj}\Lambda)$ by ${}_EX$, where ${}_EX^1 = {X'}^0$, ${}_EX^i = X^{i-1}$ for all $i \in \{2, \ldots, n+1\}$, $d_{{}_EX}^1 = {d'}^0$, $d_{{}_EX}^i = d_X^{i-1}$ for all $i \in \{2, \ldots, n\}$.

Theorem (Y. Calderón-Henao, F. Gallego-Olaya, and —)

Let Λ be an algebra with finite strong global dimension $\eta = \mathrm{s.gl.dim}(\Lambda)$. Let X, Z, and $Y = \bigoplus_{i=1}^{m} Y_i$ be with X, Z and Y_i indecomposable complexes in $\mathrm{C}_{\eta+1}(\mathrm{proj}\Lambda)$ such that neither X nor Z nor Y_i are $\mathcal{E}_{\eta+1}$-projective-injective.

[i.] Up to translation. $f : X \to Z$ is an irreducible morphism in $\mathrm{D}^b(\Lambda)$ if and only if $f : X \to Z$ is an irreducible morphism in $\mathrm{C}_{\eta+1}(\mathrm{proj}\Lambda)$, X cannot be extended to the left in $\mathrm{C}_{\eta+2}(\mathrm{proj}\Lambda)$, and Z cannot be extended to the right in $\mathrm{C}_{\eta+2}(\mathrm{proj}\Lambda)$.

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[ii.] Up to translation. $\triangle : X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is an Auslander-Reiten triangle in $\mathrm{D}^b(\Lambda)$ if and only if there exists $\delta : X \xrightarrow{\begin{bmatrix} u & u' \end{bmatrix}^t} Y \oplus P \xrightarrow{\begin{bmatrix} v' & v'' \end{bmatrix}} Z$ an almost split $\mathcal{E}_{\eta+1}$-sequence in $\mathrm{C}_{\eta+1}(\mathrm{proj}\Lambda)$ which induce an Auslander-Reiten triangle isomorphic to $\triangle$, where P is an $\mathcal{E}_{\eta+1}$-projective-injective, X cannot be extended to the left in $\mathrm{C}_{\eta+2}(\mathrm{proj}\Lambda)$, Y_i cannot be extended to the left and cannot be extended to the right in $\mathrm{C}_{\eta+2}(\mathrm{proj}\Lambda)$, and Z cannot be extended to the right in $\mathrm{C}_{\eta+2}(\mathrm{proj}\Lambda)$. Moreover, if $P \neq 0$, then P is an indecomposable complex.

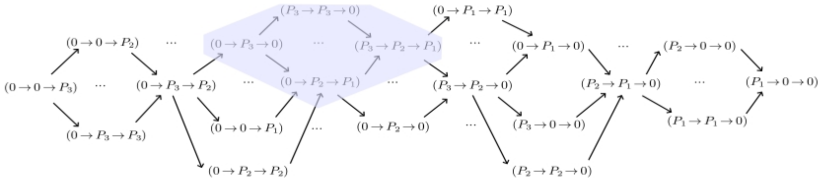
Example

Let Λ be the path algebra given by the following quiver

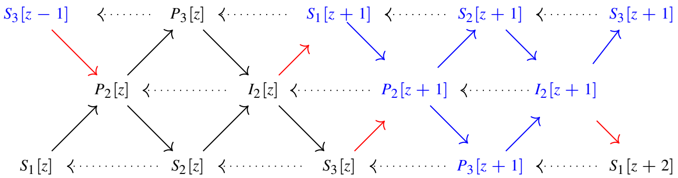
$$Q \quad : \quad 1 \quad \xrightarrow{\alpha} \quad 2 \quad \xrightarrow{\beta} \quad 3$$

with the single relation $\alpha\beta = 0$.

Auslander-Reiten Quiver of $C_3(\text{proj}\Lambda)$



The Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda\text{-mod})$ is given by



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Wild Thanks