Homological theory of representations having pure acyclic injective resolutions

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Outlines

1 Introduction

2 Preliminaries

3 Main Results

Introduction

The study of relation between representations and their components is a traditional and important research subject in the theory of homological algebra.

Introduction

- A representation X in $\text{Rep}(Q, R)$ of a right rooted quiver Q is injective iff $\psi_i^X : X(i) \to \prod_{a:i\to j} X(j)$ is a splitting epimorphism, and $X(i)$ is an injective R-module for $\forall i \in Q_0$.

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- A representation X in $\text{Rep}(Q, R)$ of a left rooted quiver Q is projective (flat) iff $\varphi_i^X : \bigoplus_{a:j\to i} X(j) \to X(i)$ is a splitting epimorphism (pure monomorphism), and $X(i)$ is a projective (flat) R-module for $\forall i \in Q_0$.

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Introduction

- An R-module L is said to be fp-injective (absolutely pure) if $\mathrm{Ext}_R^1(P,L) = 0$ for every finitely presented R-module P, or equivalently, if every exact sequence $0 \to L \to M \to N \to 0$ of R-modules is pure.
- An R-module N is flat if and only if every exact sequence $0 \to L \to M \to N \to 0$ of R-modules is pure, and so fp-injective modules are often regarded as dual analogues of flat modules.

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- An R-module L is called *strongly fp-injective* if $\mathrm{Ext}_R^n(P, L) = 0$ for any finitely presented R-module P and all $n \geq 1$.

◇ W. Li, J. Guan, B. Ouyang, Strongly FP-injective modules, *Comm. Algebra* **45**(9) (2017) 3816-3824.

Introduction

Objective:

- We were inspired to investigate strongly fp-injective representations in the category Rep(Q, R), and we will show that such representations share many nice homological properties.

Preliminaries

Let R be a ring and $Q = (Q_0, Q_1)$ a quiver.

$R\text{-Mod}$ the category of (left) R-modules.

$\text{Rep}(Q, R)$ the category of representations of Q by R-modules.

Preliminaries

Definitions:

A quiver $Q = (Q_0, Q_1)$ is a directed graph, where Q_0 and Q_1 are the sets of vertices and arrows, respectively. A representation X of Q by R-modules is a covariant functor $X : Q \to R$-Mod. Thus a representation X is determined by assigning an R-module $X(i)$ to each $i \in Q_0$ and an R-homomorphism $X(a) : X(i) \to X(j)$ to each $a \in Q_1$. A morphism $f : X \to Y$ between representations X and Y is a family of homomorphisms $\{f(i) : X(i) \to Y(i)\}_{i \in Q_0}$ such that the diagram

$$\begin{array}{ccc} X(i) & \xrightarrow{X(a)} & X(j) \\ {\scriptstyle f(i)}\downarrow & & \downarrow{\scriptstyle f(j)} \\ Y(i) & \xrightarrow{Y(a)} & Y(j) \end{array}$$

commutes for each arrow $a : i \to j$ in Q_1.

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Preliminaries

Notations:

- For a vertex $i \in Q_0$, we denote by $Q_1^{i\to *}$ (respectively $Q_1^{*\to i}$) the set of arrows in Q whose source (resp. target) is the vertex i, that is,

$$Q_1^{i\to *} := \{a \in Q_1 | s(a) = i\} \quad \text{and} \quad Q_1^{*\to i} := \{a \in Q_1 | t(a) = i\}.$$

- Let $X \in \mathrm{rep}(Q, R)$. By the universal properties of coproducts and products, there are unique homomorphisms

$$\varphi_i^X : \bigoplus_{a\in Q_1^{*\to i}} X(s(a)) \longrightarrow X(i) \text{ and } \psi_i^X : X(i) \longrightarrow \prod_{a\in Q_1^{i\to *}} X(t(a)).$$

Preliminaries

Definitions:

- By $Q^{\text{op}} = (Q_0, Q_1^{\text{op}})$ we mean a quiver with the same set of vertices and the set of reversed arrows.
- A quiver Q is called *right rooted* if there is no infinite sequence of arrows of the form $\bullet \to \bullet \to \bullet \to \cdots$ in Q. Dually, a quiver Q is *left rooted* if and only if it has no infinite sequence of arrows of the form $\cdots \to \bullet \to \bullet \to \bullet$.
- We note that a quiver Q is right rooted if and only if Q^{op} is left rooted.

Preliminaries

Tensor products of representations

Definition

An exact sequence $\eta : 0 \to X \to Y \to Z \to 0$ *of representations in Rep*(Q, R) *is called pure if* $S \otimes_Q \eta : 0 \to S \otimes_Q X \to S \otimes_Q Y \to S \otimes_Q Z \to 0$ *is exact for any representation* $S \in$ *Rep*(Q^{op}, R^{op}).

Lemma

Let $\eta : 0 \to X \to Y \to Z \to 0$ *be an exact sequence in Rep*(Q, R)*. Then* η *is pure if and only if the sequence* $\eta^+ : 0 \to Z^+ \to Y^+ \to X^+ \to 0$ *is splitting exact in Rep*(Q^{op}, R^{op}).

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Lemma

Let $\eta : 0 \to X \to Y \to Z \to 0$ be an exact sequence in Rep(Q, R). Then η is pure if and only if the sequence $\eta^+ : 0 \to Z^+ \to Y^+ \to X^+ \to 0$ is splitting exact in Rep(Q^{op}, R^{op}).

Preliminaries

Definition. Recall that a representation X is called *fp-injective*, or *absolutely pure* if every exact sequence $0 \to X \to Y$ is pure.

Lemma

Let X be a representation in Rep(Q, R). If X is fp-injective, then ψ_i^X is a pure epimorphism, and $X(i)$ is an fp-injective R-module for each $i \in Q_0$. The converse holds provided that the quiver Q is right rooted, and R is left coherent.

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Main Results

Strongly fp-injective representations

Definition. A representation X is called strongly fp-injective if it admits a pure acyclic injective resolution $0 \to X \to E^0 \to E^1 \to \cdots$ in Rep(Q, R) with E^i injective for all $i \geq 0$.

Remark

The following statements hold:

(1) *Injectives $\Rightarrow$ strongly fp-injectives $\Rightarrow$ fp-injectives.*

(2) *If $0 \to X \to E^0 \to E^1 \to \cdots$ is a pure acyclic injective resolution of X, then $\mathrm{Ker}(E^i \to E^{i+1})$ is strongly fp-injective for $i \geq 0$.*

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Strongly fp-injective representations

Theorem 1. The following statements hold for any quiver Q:

(1) The class of all strongly fp-injective representations is closed under extensions, taking cokernels of monomorphisms, finite direct sums and direct summands.

(2) Let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence of representations. If Y and Z are strongly fp-injective, then so is X.

Corollary

A representation X is strongly fp-injective iff any exact sequence $0 \to X \to E^0 \to E^1 \to \cdots$ with E^i injective for $\forall i \geq 0$ is pure.

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Theorem 2. Let X be a strongly fp-injective representation in Rep(Q, R). Then the following statements hold.

(1) For each vertex $v \in Q_0$, $X(v)$ is a strongly fp-injective R-module.

(2) For each vertex $v \in Q_0$, the homomorphism $\psi_v^X : X(v) \longrightarrow \prod_{a \in Q_1^{v \to *}} X(t(a))$ induced by $X(a) : X(v) \longrightarrow X(t(a))$ is a pure epimorphism.

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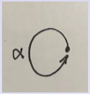
The CONVERSE of Theorem 2 is not true.



Example. Let Q be a quiver with one vertex v and an arrow α from v to itself, and the category of representations of Q by $\mathbf{k}$-vector spaces ($\mathbf{k}$ is a field). In this case Rep($Q, \mathbf{k}$) is equivalent to the category $\mathbf{k}[x]$-Mod of $\mathbf{k}[x]$-modules. If we take the representation X with $\mathbf{k}[x, x^{-1}]$ in the vertex v and the morphism $X(\alpha) : \mathbf{k}[x, x^{-1}] \to \mathbf{k}[x, x^{-1}]$ given by multiplying x. It is clear that X satisfies (1) and (2) in Theorem 2, but X is not strongly fp-injective.

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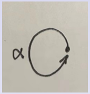

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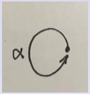

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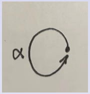

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Theorem 3. Let Q be a right rooted quiver and X a representation in $\mathrm{Rep}(Q,R)$. Then X is strongly fp-injective if and only if the following statements hold.

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Let Q be a right rooted quiver and $\mathcal{J}$ the class of strongly fp-injective modules, then

$$\Psi(\mathcal{J}) = \left\{ X \in \mathrm{Rep}(Q,R) \;\middle|\; \begin{array}{l} \psi_v^X \text{ is an epimorphism, } X(v) \text{ and} \\ \mathrm{Ker}(\psi_v^X) \in \mathcal{J} \text{ for each } v \in Q_0 \end{array} \right\}$$

is exactly the class of strongly fp-injective representations of Q.

Corollary

Let Q be right rooted. Then the following statements hold.

(1) $\Psi(\mathcal{J})$ *is closed under direct products.*

(2) $({}^{\perp}(\Psi(\mathcal{J})), \Psi(\mathcal{J}))$ *is a complete cotorsion pair in* $\mathrm{Rep}(Q,R)$.

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Thank You!