Quantum $A = U$

Based on arXiv:2407.02480

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ICRA, Shanghai, 2024.08

Outline

1. Overview
   - Backgrounds
   - Main results and comments
2. Cluster algebras
3. Cluster operations
   - Freezing Operators
   - Base Change

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- Ordinary cluster algebra (algebra definition)
  $A = \Bbbk[\text{generators}]/\text{relations}$
- Upper cluster algebra (geometry definition)
  $U = \Bbbk[\text{cluster variety } \mathcal{A}]$

Fundamental yet largely open problem

When do we have $A = U$?

Known results

- Classical level: many A from (higher) Teichmüller theory / Lie theory (see [IOS23] for a list)
- Quantum level: double Bruhat cell $G^{u,v}$ [GY20]

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Main result in this talk

Theorem [Qin24a]

For almost all the known $\boldsymbol{U}$ from Lie theory, we have $\boldsymbol{A} = \boldsymbol{U}$ at the classical & quantum levels.

Almost all the known cluster algebras from Lie theory

Classical or quantum $\Bbbk[\mathcal{A}] \simeq$ some $\boldsymbol{U}$ or $\overline{\boldsymbol{U}}$ (frozen not inverted)

- G: C finite type (in preparation)
- Subvarieties of G: $N(w)$, N^w, $G^{u,v}$
- Configurations of (partial) flags: Grassmannian, Positroids, Open Richardson, double Bott-Samelson cells, Braid varieties

$K_0(\mathscr{M})$, $\mathscr{M}$: some categories of $U_q(\widehat{\mathfrak{g}})$mod [HL10][HL13]

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  - Varieties $\mathcal{A}$, $\mathcal{A}'$ from Lie theory can be very different. But $\boldsymbol{U} = \mathbb{k}[\mathcal{A}]$, $\boldsymbol{U}' = \mathbb{k}[\mathcal{A}']$ are closely-related!
- Introduce operations relating closely-related $\boldsymbol{U}$, $\boldsymbol{U}'$
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  - Base change

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  - bases
  - quasi-categorification (monoidal categorification up to mild changes)
  - sometimes, $\boldsymbol{A} = \boldsymbol{U}$
- Extension and reduction
  - Extend the generalized Cartan matrix C to $\tilde{C}$ (Kac-Moody).
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Unexamined cases in the Theorem

- The exotic cluster structures on simple Lie groups [GSV23]
- The cluster structure on double Bott-Samelson cells by [EL21]: (Expected to equal the cluster structure [SW21] we use.)

Other applications of our approach [Qin24a]

For these cluster algebras from Lie theory, we obtain

- analogs of the dual canonical bases (common triangular bases [Qin17])
- quasi-categorification when C is symmetric.

Double Bott-Samelson cells have stronger properties:

- $\overline{A} = \overline{U}$ (frozen variables are not inverted)
- Categorified by new monoidal categories determined by positive braids, when C is of type ADE.

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- $I = I_{\mathrm{uf}} \sqcup I_{\mathrm{f}}$ unfrozen/frozen vertices, symmetrizers $d_i \in \mathbb{N}_{>0}$

A seed (chart) $\mathbf{t} = (\widetilde{B}, (x_i)_{i\in I})$:

- $\widetilde{B} = (b_{ik})$: $I \times I_{\mathrm{uf}}$ $\mathbb{Z}$-matrix, $b_{ik}d_k = -b_{ki}d_i$
- x_i: cluster variables; x_j, $j \in I_{\mathrm{f}}$: frozen variables

$\boldsymbol{LP}(\mathbf{t}) = \Bbbk[x_i^{\pm}] = \oplus_{\underline{m}\in\mathbb{Z}^I} \Bbbk x^{\underline{m}}$, where $x_i = x^{f_i}$, f_i= unit vector

- commutative product $\cdot$
- cluster monomials: $x^{\underline{m}}$, $\underline{m} \geq 0$.
- localized cluster monomials: $x^{\underline{m}}$, $\underline{m}_k \geq 0$ for $k \in I_{\mathrm{uf}}$
- $\overline{\boldsymbol{LP}}(t) = \Bbbk[x_k^{\pm}]_{k\in I_{\mathrm{uf}}}[x_j]_{j\in I_{\mathrm{f}}}$

Assume $\widetilde{B}$ is of full rank. Then there exists $d'_k \in \mathbb{N}_{>0}$ and a skew-symm form λ on $\mathbb{Z}^I$, s.t. $\lambda(f_i, \mathrm{col}_k \widetilde{B}) = -\delta_{ik} \cdot d'_k$

- Twisted product $*$ on $\boldsymbol{LP}(t)$: $x^{\underline{m}} * x^{\underline{m}'} = v^{\lambda(\underline{m},\underline{m}')} x^{\underline{m}+\underline{m}'}$

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  - $\Delta^+ :=$ {all seeds obtained from mutations}
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Two seeds for $\Bbbk[N]$, $N \subset SL_3$

$x_1' = x_1^{-1} \cdot x_2 + x_1^{-1} \cdot x_3$

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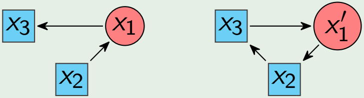

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Pointed elements and good bases

- In any seed $\mathbf{t}$, define $y^n := x^{\widetilde{B}n}$ for $n \in \mathbb{Z}^{I_{\mathrm{uf}}}$.

An element $z \in \boldsymbol{LP}(\mathbf{t})$ is $\underline{m}$-pointed if $z = x^{\underline{m}} \cdot \sum_{\underline{n} \geq 0} c_{\underline{n}} y^{\underline{n}}$, $c_0 = 1$, $c_{\underline{n}} \in \Bbbk$.

- $\deg^{\mathbf{t}} z := \underline{m}$

For $\mathrm{A} = \boldsymbol{A}$ or $\boldsymbol{U}$, a pointed basis of A is a $\Bbbk$-basis $\mathbf{S} = \{s_{\underline{m}} | \underline{m} \in \mathbb{Z}^I\}$, such that $s_{\underline{m}}$ are $\underline{m}$-pointed in $\boldsymbol{LP}(\mathbf{t})$.

A pointed basis $\mathbf{S} = \{s_{\underline{m}}\}$ is a a good basis (satisfying the Fock-Goncharov conjecture), if $\forall \mathbf{t}'$, $s_{\underline{m}}$ are $\phi_{\mathbf{t}',\mathbf{t}}\underline{m}$-pointed in $\boldsymbol{LP}(\mathbf{t}')$, where $\phi_{\mathbf{t}',\mathbf{t}}$ is the tropical mutation from $\mathbf{t}$ to $\mathbf{t}'$.

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Pointed elements and good bases

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Nice cluster decomposition

Nice cluster decomposition

A nice cluster decomposition of $z \in \boldsymbol{LP}(\mathbf{t})$ is a finite decomposition $z = \sum b_i z_i$, such that $b_i \in \mathbb{k}$, z_i are products of cluster variables of $\boldsymbol{U}$ and x_j^{-1}, $j \in I_{\mathrm{f}}$, and $\deg^{\mathbf{t}} z_i$ are different.

- If z has a nice cluster decomposition, $z \in \boldsymbol{A}$.

The dual canonical basis of $\mathbb{k}[N^w]$ is a good basis by [Qin17, Qin20]. Moreover, its elements have nice cluster decompositions (into the dual PBW basis).

Haveing a nice cluster decomposition is a technical condition depending on $\mathbf{t}$.

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Outline

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1. Overview
   - Backgrounds
   - Main results and comments
2. Cluster algebras
3. Cluster operations
   - Freezing Operators
   - Base Change

Freezing operators

- Given $F \subset I_{\mathrm{uf}}$, freeze F in $\mathbf{t} \Rightarrow$ a new seed $\mathbf{t}' = \mathfrak{f}_F \mathbf{t}$

Take any $\underline{m}$-pointed $z = x^{\underline{m}} \cdot (1 + \sum_{\underline{n}>0} c_{\underline{n}} y^{\underline{n}})$ in $\boldsymbol{LP}(\mathbf{t})$

$$\mathfrak{f}(z) := x^{\underline{m}} \cdot (1 + \sum_{\underline{n}>0} c_{\underline{n}} y^{\underline{n}})|_{y_k \mapsto 0, \forall k \in F}$$

$z = x_2^{-1} x_6 (1 + y_2 + y_2 y_1 + y_2 y_4 + y_2 y_1 y_4 + y_2 y_1 y_4 y_3)$

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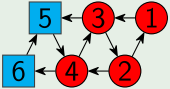

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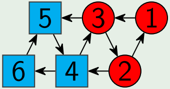

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Properties

Theorem[Qin24b]

Assume that z is a localized cluster monomial for $\boldsymbol{A}$, then $\mathfrak{f}(z)$ is a localized cluster monomial for $\boldsymbol{A}'$.

Corollary

If z has a nice cluster decomposition in $\mathbf{t}$, so does $\mathfrak{f}(z)$ in $\mathbf{t}'$.

Theorem

If $\boldsymbol{A} = \boldsymbol{U}$ and $\mathbf{S}$ is a good basis for $\boldsymbol{U}$ such that its elements have nice cluster decomposition in $\mathbf{t}$, then $\boldsymbol{A}' = \boldsymbol{U}'$ and $\mathfrak{f}(\mathbf{S})$ is a good basis for $\boldsymbol{U}'$ such that its elements have cluster decomposition in $\mathbf{t}'$.

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Outline

Similar seeds

- Assume $\mathbf{t}$ and $\mathbf{t}'$ are similar: $\widetilde{B}_{I_{\mathrm{uf}},I_{\mathrm{uf}}} = \widetilde{B}'_{I_{\mathrm{uf}},I_{\mathrm{uf}}}$ up to relabeling vertices
  - They share many properties, such as F-polynomials [FZ07]

Example

Cluster algebras on double Bruhat cells and reduced double Bruhat cells are similar.

$\Bbbk[G^{w_0,w_0}]$ has such a seed $\mathbf{t}'$

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$\Bbbk[G]$ is the associated $\overline{U}$

The reduced double Bruhat cell $\Bbbk[G^{w_0,w_0}/H]$ is a special case of double Bott-Samelson cells

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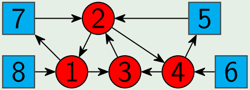

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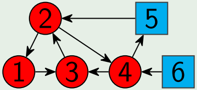

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Base change

- Take $\mathrm{A} = \boldsymbol{A}$ or $\boldsymbol{U}$. It is an algebra over the frozen torus R.
- Assume there is a homomorphism $\mathrm{var} : \boldsymbol{LP} \to \boldsymbol{LP}'$ which "preserves the seed structure" and restricts to $\mathrm{var} : R \to R'$.
  (variation map [Qin17][KQW23];
  classical case: quasi-cluster homomorphism [Fra16])

Assume A has a pointed basis $\mathbf{S} = \{s_m\}$ which factors through frozen variables: $\forall j \in I \in I_f$, $x_j \cdot s_{\underline{m}} = s_{\underline{m}+f_j}$. Then var induces $\varphi : \mathrm{A}' \simeq R' \otimes_R \mathrm{A}$, which induces a pointed basis $\mathbf{S}'$ for A'.

For $\Bbbk = \mathbb{C}$, obtain a base change (fiber product) of cluster varieties

$$\begin{array}{ccc} \mathrm{spec}\mathrm{A} & \xrightarrow{\varphi^*} & \mathrm{spec}\mathrm{A}' \\ \downarrow & & \downarrow \\ \mathrm{spec}R & \xrightarrow{\varphi^*} & \mathrm{spec}R' \end{array}$$

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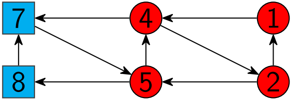
Properties

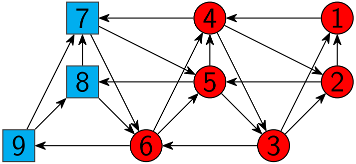
Theorem

[Qin24a] If $\boldsymbol{A} = \boldsymbol{U}$ and $\boldsymbol{U}$ has a good basis $\mathbf{S}$, such that the basis element have nice cluster decomposition in $\mathbf{t}$, then via the base change φ, we have $\boldsymbol{A}' = \boldsymbol{U}'$.
Moreover, $\mathbf{S}'$ is a good basis of $\boldsymbol{U}'$ and its basis elements have nice cluster decomposition in $\mathbf{t}'$.

Appendix: $K_0(U_q(\widehat{\mathfrak{g}})\text{mod})$ from $\widetilde{N}^{\widetilde{w}}$



$\mathbf{t}'$ for $\mathscr{C}_2$ [HL10], type A_2
(subcategory of $U_q(\widehat{\mathfrak{sl}_3})$ mod)

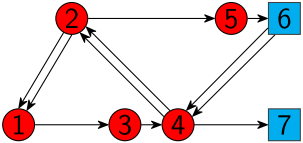


$\mathbf{t}$ for $\Bbbk[\widetilde{N}^{c^3}]$, type $A_2^{(1)}$
$c = s_1 s_2 s_3$ [GLS11]

$\mathbf{t}$ becomes $\mathbf{t}'$ after the following procedure:

1. Freeze $3, 6$,
2. Remove the frozen $3, 6, 9$ (base change)

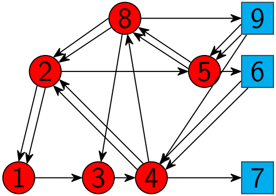
Appendix: Double Bott-Samelson Cells from $\widetilde{N}^{\widetilde{w}}$



$\mathbb{k}[\mathrm{dBS}]$ has such a seed $\mathbf{t}'$

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\underline{i}' = (1,2,1,1,2,2,1)$$



$\mathbf{t}$ for $\mathbb{k}[\widetilde{N}^{\widetilde{w}}]$

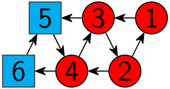
$$\widetilde{C} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}$$

$$\widetilde{w} = (1,2,1,3,1,2,3,2,1)$$

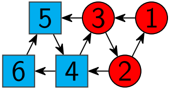
$\mathbf{t}$ becomes $\mathbf{t}'$ after the following procedure:

1. Freeze 8
2. Remove the frozen 8, 9 (base change)

Interpretation



$z = x_2^{-1} x_6 (1 + y_2 + y_2 y_1 + y_2 y_4 + y_2 y_1 y_4 + y_2 y_1 y_4 y_3)$



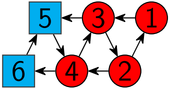
$\mathfrak{f}(z) = x_2^{-1} x_6 (1 + y_2 + y_2 y_1)$

In cluster categories (2-Calabi-Yau triangulated categories)

Let $\mathscr{C}$ be the cluster category for $\mathbf{t}$, CC( ) cluster character, $\mathrm{CC}(T_i) = x_i(\mathbf{t})$. Freezing 4:

- Restrict to the subcategory $\mathscr{C}' = \{V' \in \mathscr{C} \,|\, \mathrm{Hom}_{\mathscr{C}}(V', T_4[1]) = 0\}$
- $\mathfrak{f}(\mathrm{CC}(V)) = \mathrm{CC}(V') \cdot x_4^{-N}$, $N \gg 1$,
  - V' is a generic extension of V and $T_4^{\oplus N}$: $T_4^{\oplus N} \to V' \to V \to T_4^{\oplus N}[1]$
- $\mathrm{Hom}(T, V')$ is the maximal submodule of $\mathrm{Hom}(T, V)$ such that $(\dim \mathrm{Hom}(T, V'))_4 = 0$.

Interpretation



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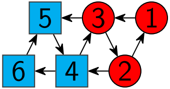

$\mathfrak{f}(z) = x_2^{-1} x_6 (1 + y_2 + y_2 y_1)$

In monoidal categories with good properties

Assume $\boldsymbol{U} \simeq K_0(\mathscr{M})$, $x_i(\mathbf{t}) = [S_i]$. Freezing 4:

- Restrict to the subcategory $\mathscr{M}'$ generated by simples $S' \in \mathscr{M}' : S' \otimes S_4$ remains simple
- $\forall$ simple $S \in \mathscr{M}$, $\mathfrak{f}([S]) = [S'] \cdot x_4(\mathbf{t})^{-N}$, $N \gg 1$
  - S' is the simple head of $S \otimes S_4^{\otimes N}$

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