On relative Koszul duality and dg enhanced orbit categories

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Outline

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Our goals

Goal 1. Give a general construction of (pre)triangulated hull of (dg) orbit categories.
Goal 2. Generalize Ikeda-Qiu and Happel's results respectively.
Goal 3. Connect them via relative Koszul duality.

Cluster theory

- Let A be a f.d. hereditary algebra over a field $\mathbf{k}$.
- Cluster category (Buan-Marsh-Reineke-Reiten-Todorov) as the orbit category
$$\mathcal{C}_2(A) \coloneqq \mathcal{D}^b(\operatorname{mod} A)/\tau^{-1} \circ [1],$$
where τ is the AR-translation functor.
- Additive categorification of cluster algebras (Fomin-Zelevinsky) via cluster tilting theory.

Higher cluster categories without the hereditary assumptions

- Let A be a f.d. $\mathbf{k}$-algebra with $\operatorname{gldim} A < \infty$ and $m \geqslant 2$ an integer.
- $\operatorname{per} A$ admits a Serre functor $\mathbb{S} \coloneqq - \overset{L}{\otimes}_A DA$ and set $\Sigma_m \coloneqq \mathbb{S} \circ [-m]$, which is an automorphism of $\operatorname{per} A$.
- The orbit category $\operatorname{per} A/\Sigma_m$ has a triangulated hull (Keller)

$$\mathcal{C}_m(A) \coloneqq \langle A \rangle_B / \operatorname{per} B,$$

  where B is the dg algebra $A \oplus DA[-m-1]$.
- The m-cluster category $\mathcal{C}_m(A)$ of A coincides with $\operatorname{per} A/\Sigma_m$ when A is hereditary.

Goal 1. Give a general construction of (pre)triangulated hull of (dg) orbit categories.

Amiot-Guo-Keller defined a generalized version of cluster category.

- For an integer N, the N-Calabi-Yau completion of A is the derived dg tensor algebra
$$\Pi_N A = T_A(\Theta) = A \oplus \Theta \oplus \left(\Theta \otimes_A \Theta\right) \oplus \cdots ,$$
where $\Theta = \theta[N-1]$, θ is a cofibrant replacement of $\mathrm{RHom}_{A^e}(A, A^e)$ and $A^e = A^{op} \otimes_{\mathsf{k}} A$.
- $\mathcal{C}(\Pi_N A) \coloneqq \mathrm{per}(\Pi_N A)/\,\mathrm{pvd}(\Pi_N A)$, the generalized $(N-1)$-cluster category.
- $\mathrm{pvd}(\Pi_N A)$ is N-Calabi-Yau and if $\mathcal{C}(\Pi_N A)$ is Hom-finite, then there is a triangle equivalence
$$\mathcal{C}(\Pi_N A) \simeq \mathcal{C}_{N-1}(A).$$
Furthermore, $\mathcal{C}(\Pi_N A)$ is $(N-1)$-Calabi-Yau.

$\mathbb{X}$-Calabi-Yau completion and ∞-cluster category

- Let A be a connective, smooth and proper dg algebra.
- The $\mathbb{X}$-Calabi-Yau completion $\Pi_{\mathbb{X}}(A)$ of A is a dbg algebra

$$T_A(\Theta) = A \oplus \Theta \oplus (\Theta \otimes_A \Theta) \oplus \cdots$$

  for $\Theta = \theta[\mathbb{X} - 1]$.
- $\mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) \coloneqq \operatorname{per}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) / \operatorname{pvd}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A)$, the ∞-cluster category.

Theorem (Keller, Ikeda-Qiu)

$\operatorname{pvd}^{\mathbb{Z}}(\Pi_{\mathbb{X}}(A))$ *is* $\mathbb{X}$*-Calabi-Yau and* $\mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A)$ *is* $(\mathbb{X} - 1)$*-Calabi-Yau.*

Perfect derived category as ∞-cluster category

Theorem (Ikeda-Qiu)

We have a triangle equivalence

$$\begin{array}{ccc} \mathbb{S} =? \overset{L}{\otimes}_A DA & \longrightarrow & [\mathbb{X}-1] \\ \circlearrowright & & \circlearrowright \\ \operatorname{per} A & \xrightarrow{\sim} & \mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A). \end{array} \tag{1}$$

Note that $\mathbb{S} = \tau^{-1} \circ [1]$ *is the Serre functor.*

Goal 2. Generalize Ikeda-Qiu's results above.

Perfect derived category as singularity categories

- Let A be a f.d. algebra with $\operatorname{gldim} A < \infty$.
- Let $E_{\mathbb{X}} = A \oplus DA[-\mathbb{X}]$ be the trivial extension of A.
- $\operatorname{mod}^{\mathbb{Z}}(E_{\mathbb{X}})$ is equivalent to $\operatorname{mod}(\hat{A})$, where $\hat{A}$ is the repetitive algebra.

Theorem (Happel, Rickard)

There is an equivalence

$$
\begin{array}{ccc}
\mathbb{S} \circ [1] & \longrightarrow & [\mathbb{X}] \\
\circlearrowright & & \circlearrowright \\
\operatorname{per} A & \xrightarrow{\sim} & \underline{\operatorname{mod}}^{\mathbb{Z}}(E_{\mathbb{X}}) \hookrightarrow \operatorname{sg}^{\mathbb{Z}}(E_{\mathbb{X}}) \coloneqq \operatorname{pvd}^{\mathbb{Z}}(E_{\mathbb{X}}) / \operatorname{per}^{\mathbb{Z}}(E_{\mathbb{X}}).
\end{array}
\tag{2}
$$

Goal 3. Generalize Happel's results above and build a connection with Ikeda-Qiu's results.

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Differential bigraded categories/algebras

- Let $\mathbf{k}$ be an algebraically closed field.
- Let $\mathrm{Grm}(\mathbf{k})$ be the category of $\mathbb{Z}$-graded $\mathbf{k}$-modules $M = \oplus_{p\in\mathbb{Z}} M_p$ with grading shift given by $(M[\mathbb{X}])_p = M_{p+1}$.
- Let $\mathcal{C}_{dbg}(\mathbf{k})$ be the dg category of complexes over $\mathrm{Grm}(\mathbf{k})$.
- The differential bigraded (dbg) category $\mathcal{A}$ is a category enriched in $\mathcal{C}_{dbg}(\mathbf{k})$.

Remark

If $\mathcal{A}$ is a dbg category with one object $*$, then we can identity $\mathcal{A}$ with the dbg algebra $A = \mathcal{A}(*,*)$, i.e. a dg $\mathbf{k}$-algebra endowed with the Adams $\mathbb{Z}$-grading

$$A = \oplus_{p\in\mathbb{Z}} A_p$$

such that $|d| = (1,0)$.

The derived categories

The bigraded derived category $\mathcal{D}^{\mathbb{Z}}(\mathcal{A})$ is the category of dbg modules $M : \mathcal{A}^{op} \to \mathcal{C}_{dbg}(\mathbf{k})$ localized at the $s : M \to L$ such that $s_p X : L_p X \to M_p X$ is a quasi-isomorphism for any $p \in \mathbb{Z}$ and $X \in \mathrm{obj}(\mathcal{A})$.

Remark

$\mathcal{D}^{\mathbb{Z}}(\mathcal{A})$ is compactly generated by $\{X^\wedge[p\mathbb{X}], p \in \mathbb{Z}, X \in \mathrm{obj}(\mathcal{A})\}$, where $X^\wedge := \mathcal{A}(?, X)$.

The perfect derived category is

$$\mathrm{per}^{\mathbb{Z}}(\mathcal{A}) := \mathrm{thick}\{X^\wedge[p\mathbb{X}], p \in \mathbb{Z}, X \in \mathrm{obj}(\mathcal{A})\}.$$

The perfectly valued derived category is

$$\mathrm{pvd}^{\mathbb{Z}}(\mathcal{A}) := \{M \in \mathcal{D}^{\mathbb{Z}}(\mathcal{A}) \mid MX \in \mathrm{per}^{\mathbb{Z}}(\mathbf{K}), \forall X \in \mathrm{obj}(\mathcal{A})\},$$

where $\mathbf{K}$ is $\mathbf{k}$ when regarded as a dbg algebra with trivial grading shift.

Let A be a dg algebra, B a dbg algebra and $i_A^B : A \to B$ a morphism of dbg algebras. There is an induced restriction functor:

$$\mathcal{D}^{\mathbb{Z}}(B) \to \mathcal{D}^{\mathbb{Z}}(A).$$

Definition

We define the relative perfectly valued derived categories of B, with respect to A, to be

$$\mathrm{pvd}^{\mathbb{Z}}(B, A) \coloneqq \{M \in \mathcal{D}^{\mathbb{Z}}(B) \big| M|_A \in \mathrm{per}^{\mathbb{Z}}(A)\}. \tag{3}$$

Let $X \in \mathcal{D}(A^e)$ an invertible dg bimodule with inverse Y, i.e.

$$X \overset{L}{\otimes}_A Y \simeq A \text{ and } Y \overset{L}{\otimes}_A X \simeq A$$

in $\mathcal{D}(A^e)$.

Write $\widehat{X}$ for a cofibrant resolution of $X[\mathbb{X}-1]$ and let

$$\mathsf{T} = T_A(\widehat{X}) = \bigoplus_{p\geqslant 0} \widehat{X}^{\otimes_A^p} \tag{4}$$

be the differential bigraded tensor algebra of $\widehat{X}$ over A and

$$\mathsf{E} = A \oplus Y[-\mathbb{X}] \tag{5}$$

the differential bigraded trivial extension algebra of $Y[-\mathbb{X}]$ by A.

Remark

The relative perfectly valued derived categories $\mathrm{pvd}^{\mathbb{Z}}(\mathsf{T}, A)$ *and* $\mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A)$ *equal the thick subcategory of* $\mathcal{D}^{\mathbb{Z}}(\mathsf{T})$ *and* $\mathcal{D}^{\mathbb{Z}}(\mathsf{E})$ *generated by the* $A[q\mathbb{X}], q \in \mathbb{Z}$ *respectively.*

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Enlarged cluster categories

Definition

We define the enlarged cluster category of A wrt X as the Verdier quotient

$$\mathcal{C}^{\mathbb{Z}}(\mathrm{T}, A) \coloneqq \mathrm{per}^{\mathbb{Z}}(\mathrm{T}) / \mathrm{pvd}^{\mathbb{Z}}(\mathrm{T}, A). \tag{6}$$

A generalization of Ikeda-Qiu:

Theorem (F-Keller-Qiu)

The composition

$$\begin{array}{cccccc} & ?\overset{L}{\otimes}_A Y[1] & & \longrightarrow & & [\mathbb{X}] \\ & \circlearrowright & & & & \circlearrowright \\ \Phi : & \mathrm{per}\,A & \xrightarrow{?\overset{L}{\otimes}_A \mathrm{T}} & \mathrm{per}^{\mathbb{Z}}(\mathrm{T}) & \longrightarrow & \mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) \end{array} \tag{7}$$

is a triangle equivalence, where $[\mathbb{X}] \circ \Phi \simeq \Phi \circ (? \overset{L}{\otimes}_A Y[1])$.

Shrunk singularity categories

Definition

We define the <u>shrunk singularity category</u> of A as the Verdier quotient

$$\mathrm{sg}^{\mathbb{Z}}(\mathsf{E}, A) \coloneqq \mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A)/\,\mathrm{per}^{\mathbb{Z}}(\mathsf{E}). \tag{8}$$

A generalization of Happel:

Theorem (Hanihara, F-Keller-Qiu)

The restriction along the augmentation $E \to A$ induces an equivalence

$$\begin{array}{ccc} ? \overset{L}{\otimes}_A Y[1] \circlearrowright & \longrightarrow & [\mathbb{X}] \circlearrowright \\ \Psi : \ \mathrm{per}\, A & \xrightarrow{\ \sim\ } & \mathrm{sg}^{\mathbb{Z}}(\mathsf{E}, A) \end{array} \tag{9}$$

where $[\mathbb{X}] \circ \Psi \simeq \Psi \circ (? \overset{L}{\otimes}_A Y[1])$.

The relative Koszul duality

Theorem (F-Keller-Qiu)

The adjoint pair

$$\mathrm{per}^{\mathbb{Z}}(\mathsf{T}) \underset{?\overset{L}{\otimes}_{\mathsf{E}}A}{\overset{\mathrm{RHom}^{\mathbb{Z}}_{\mathsf{T}}(A,?)}{\rightleftarrows}} \mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A). \tag{10}$$

induces the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{pvd}^{\mathbb{Z}}(\mathsf{T}, A) & \hookrightarrow & \mathrm{per}^{\mathbb{Z}}(\mathsf{T}) & \longrightarrow & \mathcal{C}^{\mathbb{Z}}(\mathsf{T}, A) & \xleftarrow{[1]\circ\Phi,\sim} & \\
\downarrow \wr & & \downarrow \wr.\ \mathrm{RHom}^{\mathbb{Z}}_{\mathsf{T}}(A,?) & & \downarrow \wr & & \mathrm{per}\, A, \\
\mathrm{per}^{\mathbb{Z}}(\mathsf{E}) & \hookrightarrow & \mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A) & \longrightarrow & \mathrm{sg}^{\mathbb{Z}}(\mathsf{E}, A) & \xleftarrow{\Psi,\sim} &
\end{array} \tag{11}$$

where "all" functors commute with $[\mathbb{X}]$.

Examples

- **Classical Koszul duality.** For the case when $A = \mathbf{k}$, $X = \mathbf{k}[1-\mathbb{X}]$ and $Y = \mathbf{k}[\mathbb{X}-1]$, we have

$$\mathsf{T} = T_A(\mathbf{k}) = \mathbf{k}[u],$$

where u is of bidegree $(0,0)$ and

$$\mathsf{E} = \mathbf{k} \oplus \mathbf{k}[-1] = \mathbf{k}[v]/(v^2),$$

where v is of bidegree $(0,0)$. We see that T and E are Koszul dual to each other in the sense that

$$\mathrm{RHom}_{\mathsf{E}}(A,A) \simeq \mathsf{T} \text{ and } \mathrm{RHom}_{\mathsf{T}}(A,A) \simeq \mathsf{E}.$$

- **The Calabi-Yau-X case.** If A is a smooth, proper and connective dg algebra, then we can take

$$X = A^{\vee} \text{ and } Y = DA,$$

In this case, $\mathsf{T} = \Pi_{\mathbb{X}} A$ and $E = E_{\mathbb{X}}$ are the $\mathbb{X}$-Calabi-Yau completion and the trivial extension of A respectively. Moreover, $\mathcal{C}^{\mathbb{Z}}(\mathsf{T}, A)$ reduces to the ∞-cluster category $\mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A)$ and $\mathrm{sg}^{\mathbb{Z}}(\mathsf{E}, A)$ becomes $\mathrm{sg}^{\mathbb{Z}}(E_{\mathbb{X}})$.

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The orbit categories

Let $\mathcal{A}$ be a dg category and $F \in \text{rep}_{dg}(\mathcal{A}, \mathcal{A})$ a dg bimodule.
The left lax quotient $\mathcal{A}/_{ll}F^{\mathbb{N}}$ of $\mathcal{A}$ by F is the dg category whose

- objects are the same as the objects of $\mathcal{A}$, and
- morphisms are given by $\mathcal{A}/_{ll}F^{\mathbb{N}}(X, Y) = \bigoplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical dg functor $Q_{\mathbb{N}} : \mathcal{A} \to \mathcal{A}/_{ll}F^{\mathbb{N}}$ acts on

- objects: it sends $X \in \text{obj}(\mathcal{A})$ to $X \in \text{obj}(\mathcal{A}/_{ll}F^{\mathbb{N}})$, and
- morphisms: it sends $f : X \to Y$ to $f \in \mathcal{A}(X, Y) \subseteq \oplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical morphism of dg functors $q : Q_{\mathbb{N}}F \to Q_{\mathbb{N}}$ acts on the objects of $\mathcal{A}$ as

$$qX \coloneqq \text{id}_{FX} \in \mathcal{A}(FX, FX) \subseteq \bigoplus_{p \in \mathbb{N}} \mathcal{A}(FX, F^{p+1}X).$$

Definition

The dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$ is defined to be the dg localization $(\mathcal{A}/_{ll}F^{\mathbb{N}})[q^{-1}]$ of $\mathcal{A}/_{ll}F^{\mathbb{N}}$ wrt the morphisms $qX : Q_{\mathbb{N}}FX \to Q_{\mathbb{N}}X$ for any $X \in \text{obj}(\mathcal{A})$.

The orbit categories with dg enhancement

The <u>$\mathbb{Z}$-equivariant category</u> $\mathbb{Z}\text{-}\operatorname{Eq}(\mathcal{A}, F, \mathcal{B})$ consists of $\mathbb{Z}$-equivariant functors from $(\mathcal{A}, F)$ to $(\mathcal{B}, \mathrm{id}_{\mathcal{B}})$ whose objects are the pairs (G, γ), where $G \in \operatorname{rep}_{dg}(\mathcal{A}, \mathcal{B})$ and $\gamma : GF \to G$ s.t. γX is an isom in $H^0(\mathcal{B}), \forall X \in \operatorname{obj}(\mathcal{A})$.

Theorem (F-Keller-Qiu)

Let $\mathcal{B}$ be a pretriangulated dg category. Then $\mathcal{A} \to \operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}})$ induces an isomorphism

$$\operatorname{rep}_{dg}(\operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}}), \mathcal{B}) \xrightarrow{\sim} \mathbb{Z}\text{-}\operatorname{Eq}(\mathcal{A}, F, \mathcal{B}). \tag{12}$$

in Hqe.

Definition

Let $\mathcal{T}$ be a triangulated category endowed with a dg enhancement $H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$ and $F \in \operatorname{rep}_{dg}(\mathcal{A}, \mathcal{A})$ be a dg bimodule. If the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ is an equivalence, then the <u>triangulated orbit category</u> of $\mathcal{T}$ wrt $\mathcal{A}$ is defined as

$$H^0(\operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}})).$$

Commuting with dg quotient

Corollary

Suppose $\mathcal{N} \subseteq \mathcal{A}$ *is a full dg subcategory such that* $H^0(\mathcal{N})$ *is stable under* $H^0(F)$*, so that* F *induces dg bimodules* $F_{\mathcal{N}} \in \mathrm{rep}_{dg}(\mathcal{N}, \mathcal{N})$ *and* $F_{\mathcal{A}/\mathcal{N}} \in \mathrm{rep}_{dg}(\mathcal{A}/\mathcal{N}, \mathcal{A}/\mathcal{N})$*. We have a canonical isomorphism in* Hqe

$$\mathrm{pretr}((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}}) \simeq \mathrm{pretr}(\mathcal{A}/F^{\mathbb{Z}})/\,\mathrm{pretr}(\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}}). \tag{13}$$

Moreover, we have a short exact sequence of triangulated categories

$$0 \to (\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}})^{tr} \to (\mathcal{A}/F^{\mathbb{Z}})^{tr} \to ((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}})^{tr} \to 0, \tag{14}$$

where $(-)^{tr}$ *denotes the functor* $H^0 \circ \mathrm{pretr}$.

A conjecture of Ikeda-Qiu

Let $N \geqslant 3$ be an integer. There is a projection $\pi_N \colon \Pi_{\mathbb{X}} A \to \Pi_N A$ collapsing the double degree $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ into $a + bN \in \mathbb{Z}$, that induces a functor

$$\pi_N \colon \operatorname{per}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) \to \operatorname{per}(\Pi_N A)$$

which restricts to $\pi_N \colon \operatorname{pvd}^{\mathbb{Z}} \Pi_{\mathbb{X}} A \to \operatorname{pvd}(\Pi_N A)$.

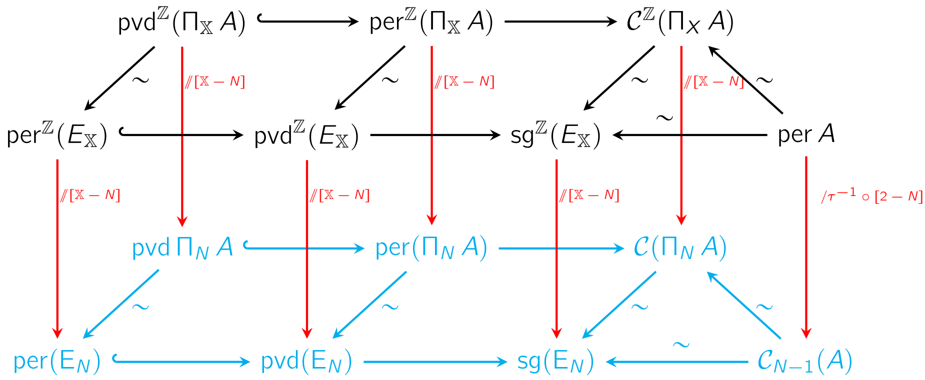
Theorem (Ikeda-Qiu, F-Keller-Qiu)

Let A be a connective, smooth and proper dg algebra. We have the following commutative diagram between short exact sequences of triangulated categories:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{pvd}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) & \longrightarrow & \operatorname{per}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) & \longrightarrow & \operatorname{per} A & \longrightarrow & 0 \\ & & \downarrow /\!\!/[\mathbb{X}-N] & & \downarrow /\!\!/[\mathbb{X}-N] & & \downarrow /\tau^{-1}\circ[2-N] & & \\ 0 & \longrightarrow & \operatorname{pvd}(\Pi_N A) & \longrightarrow & \operatorname{per}(\Pi_N A) & \longrightarrow & \mathcal{C}_{N-1}(A) & \longrightarrow & 0. \end{array} \tag{15}$$

Dually, we also have a projection $\pi_N : \mathrm{E}_{\mathbb{X}} \to \mathrm{E}_N$ that induces functors between corresponding pvd and per, where $\mathrm{E}_{\mathbb{X}} = A \oplus DA[-\mathbb{X}]$ and $\mathrm{E}_N = A \oplus DA[-N]$.

Combining everything



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Thank you!