Classifying n-representation infinite algebras of type $\tilde{A}$

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Plan

Outline

1. Overexplain the classical case
2. The higher case works the same
3. Run out of time (otherwise: Mutation lattices)

Classical type $\tilde{A}$

Let $n = 1$.

Recall

Λ with $\mathrm{gdim}(\Lambda) = 1$ is of type $\tilde{A}_m$ if $\Lambda \simeq kQ$, where Q is an acyclic orientation of the Euclidean diagram $\tilde{A}_m$.

Example

$Q =$

Classical type $\tilde{A}$

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Example

$$Q = \begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \longrightarrow & 3, \end{array} \qquad \begin{array}{ccc} & 2 & \\ \nearrow & & \nwarrow \\ 1 & \longrightarrow & 3 \end{array}$$

"Problem"

How many different orientations are there? Which ones are equivalent? How equivalent?

Preprojective gradings

Theorem/Definition [Baer-Geigle-Lenzing]

Let $\operatorname{gdim}(\Lambda) = 1$. Then the preprojective algebra $\Pi(\Lambda)$ is the *graded* algebra

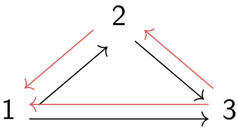
$$\Pi(\Lambda) = T_\Lambda \operatorname{Ext}^1_\Lambda(D(\Lambda), \Lambda).$$

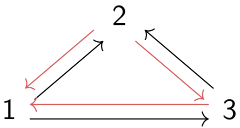
This way,

$$\text{RI algebras } \Lambda \leftrightarrow \text{“nice” gradings on } \Pi(\Lambda)$$

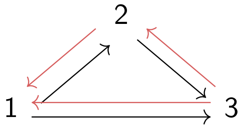
Example continued

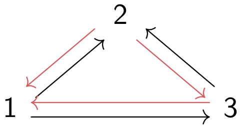
Quiver for $\Pi(kQ)$ is the double $\overline{Q}$:





Gradings, Cuts, Types





Definition

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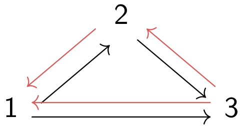

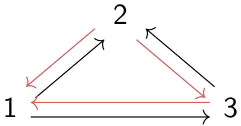

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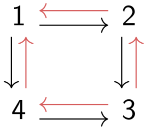
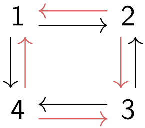
Definition

For a cut C we call $\theta(C) = (\# \circlearrowright \text{ arrows in } C, \# \circlearrowleft \text{ arrows in } C)$ its type.

Mutation

Example





Different cuts, same type. Related by reflection functors!

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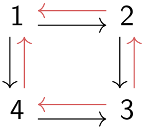

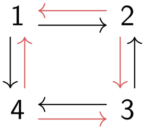

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Theorem

Two cuts have the same type iff they are related by a sequence of mutations.

Preprojective algebras, again

From McKay correspondence

Λ is of type $\tilde{A}_{m-1}$ iff $\Pi(\Lambda) = k[x, y] * C_m$, where $C_m \leq \mathrm{SL}_2(k)$.

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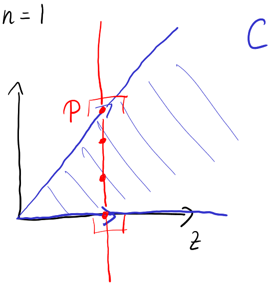
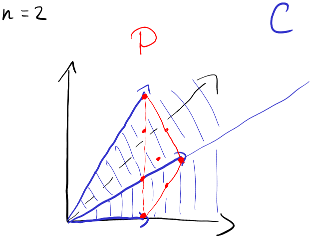
Toric varieties

If G is abelian, $X = \mathrm{Spec}(k[x_1, \ldots, x_{n+1}]^G)$ is a toric variety. This is defined by a lattice cone $C \subseteq N_{\mathbb{R}}$. Fix the "slice at height 1" $P = C \cap \{z = 1\}$.





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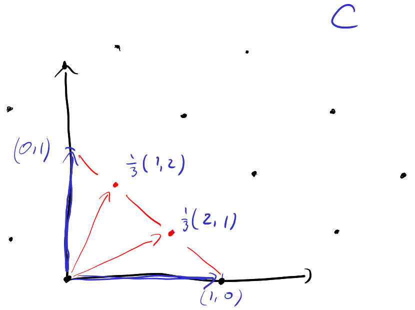
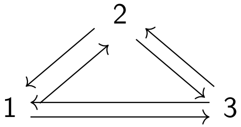
Example

$G = \langle \frac{1}{3}(1,2) \rangle$



Types

Example



We have two possible types for preprojective gradings. We also add the infinite-dimensional ones

$$(3,0) - (2,1) - (1,2) - (0,3)$$

Observation

This is precisely P!

The main result

Theorem

1. A cut on $\overline{Q}$ defines a preprojective grading iff its type is an *internal* lattice point in P.
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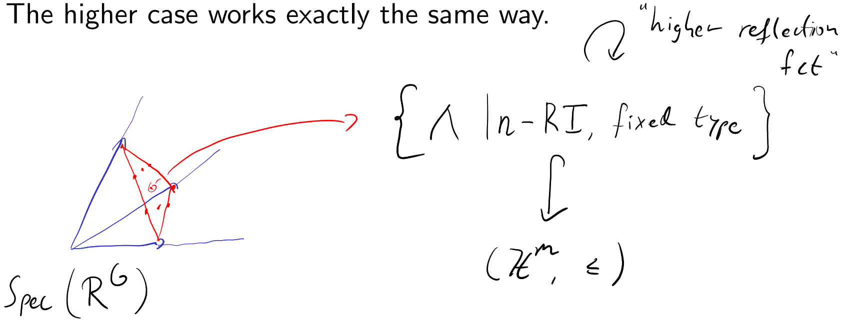
Theorem

The higher case works exactly the same way.



The higher case

Definition [Herschend-Iyama-Oppermann '14]

1. A f.d. algebra Λ is n-representation infinite (n-RI) if $\mathrm{gdim}(\Lambda) \leq n$ and
$$\nu_n^{-i}(\Lambda) \in \operatorname{mod} \Lambda$$
for all $i \geq 0$, where $\nu_n^{-1} = [n] \circ \mathbb{R}\mathrm{Hom}(D(\Lambda), -)$ the "inverse derived higher AR-translate".

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$$\Pi_{n+1}(\Lambda) := T_\Lambda \operatorname{Ext}^n_\Lambda(D(\Lambda), \Lambda) \simeq k[x_1, \ldots, x_{n+1}] * G,$$
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Problem

Which G appear? In how many ways?

Quivers for $\Pi_{n+1}(\Lambda)$

Construction

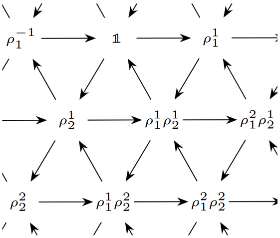
$R * G$ is basic, its quiver Q is a *Cayley-graph* of $\hat{G} = \mathrm{Hom}(G, k^*)$ wrt. $(n+1)$ generators $\{\rho_1, \ldots, \rho_{n+1}\}$. Every arrow $\alpha \in Q_1$ corresponds to some $\cdot\rho_i$. We call $\theta(\alpha) = i$ its *type*. We identify "nice" gradings on $R * G$ with cuts $C \subseteq Q_1$.

Example

$n = 1$:

$$\rho_2 \underset{\cdot\rho_2}{\overset{\cdot\rho_1}{\rightleftarrows}} 1 \underset{\cdot\rho_2}{\overset{\cdot\rho_1}{\rightleftarrows}} \rho_1 \underset{\cdot\rho_2}{\overset{\cdot\rho_1}{\rightleftarrows}} \cdots \rightleftarrows \rho_2$$

$n = 2$:



Mutation, again

Let $C \subseteq Q_1$ be a cut, $Q_C = (Q_0, Q_1 - C)$.

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The type of C is $\theta(C) = (\#\{\alpha \in C \mid \theta(\alpha) = i\})_{1 \leq i \leq n+1}$.

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If $s \in Q_C$ is a source (resp. sink), produce a new cut $\mu_s(C)$ by turning s into a sink (resp. source) in $Q_{\mu_s(C)}$.

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Note

Cut mutation $\leftrightarrow$ n-APR tilting of n-RI algebras

Types, again

$X = \mathrm{Spec}(R^G)$ toric, with cone $C \subseteq N_{\mathbb{R}}$, $P = C \cap \{z = 1\}$.

The toric argument

1. $Z(R * G) = R^G$.
2. Gradings on $R * G$ induce gradings on R^G
3. Gradings on $R^G \leftrightarrow k^*$-actions on $X \leftrightarrow$ 1-param. sbgrps. of X $\leftrightarrow$ points in N.

Theorem [D-Gasanova]

Two cuts on $R * G$ have the same type iff they induce the same grading on R^G. There's a bijection

$$\{\text{types of cuts}\} \leftrightarrow \{\text{lattice points in } P\},$$

giving *preprojective cuts* $\leftrightarrow$ *internal points* in P.

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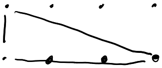
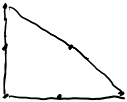
Problem

When does an internal point exist?

$n = 2$





Convex Geometry to the rescue (almost)

The set $P = C \cap \{z = 1\}$ is a lattice polytope.

Definitipn

A lattice polytope is called *hollow* if it has no interior points.

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Corollary

"Almost all" $R * G$ have a higher preprojective structure.

The mutation lattice

Recall that $Q_0 = \mathrm{Irr}(G)$. Fix the trivial representation $1 \in Q_0$. Let $M(C) = \{C' \text{ cut } \mid \theta(C) = \theta(C')\}$.

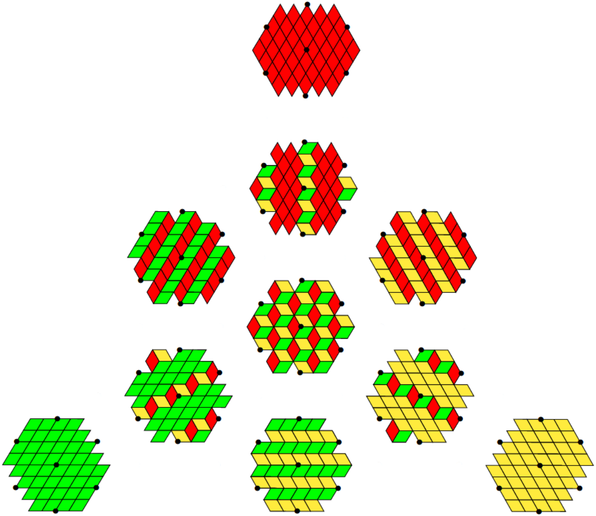
Construction

For $C_1, C_2 \in M(C)$, define $C_1 \leq C_2$ if there exists a sequence of source mutations taking C_1 to C_2, not mutating at 1. Then $(M(C), \leq)$ is a finite distributive lattice.

Proof sketch

Associate to each C_i a height function. This is an integer valued function on Q_0, and in this way $M(C)$ becomes a sublattice of $(\mathbb{Z}^m, \leq)$.

Thank you



Thank you!

Secret slide