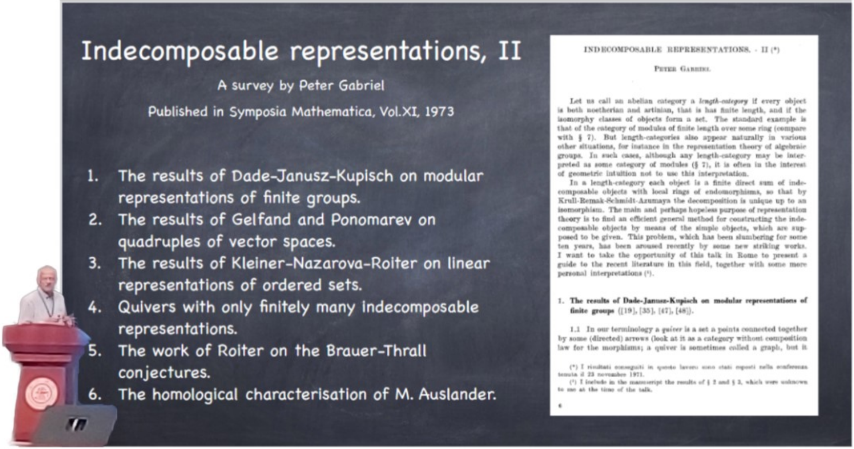
Preprojective component in a suitable Krull-Schmidt category

Ivon Dorado
iadoradoc@unal.edu.co

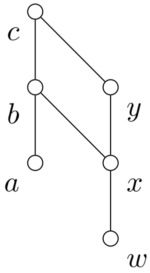
Universidad Nacional de Colombia

ICRA 21
August 6th 2024

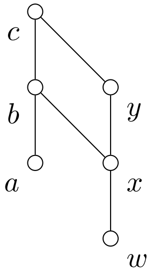

Indecomposable representations, II
A survey by Peter Gabriel
Published in Symposia Mathematica, Vol.XI, 1973
1. The results of Dade-Janusz-Kupisch on modular representations of finite groups.
2. The results of Gelfand and Ponomarev on quadruples of vector spaces.
3. The results of Kleiner-Nazarova-Roiter on linear representations of ordered sets.
4. Quivers with only finitely many indecomposable representations.
5. The work of Roiter on the Brauer-Thrall conjectures.
6. The homological characterisation of M. Auslander.
INDECOMPOSABLE REPRESENTATIONS. - II (*)
PETER GABRIEL
Let us call an abelian category a *length-category* if every object is both noetherian and artinian, that is has finite length, and if the isomorphy classes of objects form a set. The standard example is that of the category of modules of finite length over some ring (compare with § 7). But length-categories also appear naturally in various other situations, for instance in the representation theory of algebraic groups. In such cases, although any length-category may be interpreted as some category of modules (§ 7), it is often in the interest of geometric intuition not to use this interpretation.
In a length-category each object is a finite direct sum of indecomposable objects with local rings of endomorphisms, so that by Krull-Remak-Schmidt-Azumaya the decomposition is unique up to an isomorphism. The main and perhaps hopeless purpose of representation theory is to find an efficient general method for constructing the indecomposable objects by means of the simple objects, which are supposed to be given. This problem, which has been slumbering for some ten years, has been aroused recently by some new striking works. I want to take the opportunity of this talk in Rome to present a guide to the recent literature in this field, together with some more personal interpretations (1).
1. **The results of Dade-Janusz-Kupisch on modular representations of finite groups** ([19], [35], [47], [48]).
1.1 In our terminology a *quiver* is a set a points connected together by some (directed) arrows (look at it as a category without composition law for the morphisms; a quiver is sometimes called a graph, but it
(*) I risultati conseguiti in questo lavoro sono stati esposti nella conferenza tenuta il 23 novembre 1971.
(1) I include in the manuscript the results of § 2 and § 3, which were unknown to me at the time of the talk.
6

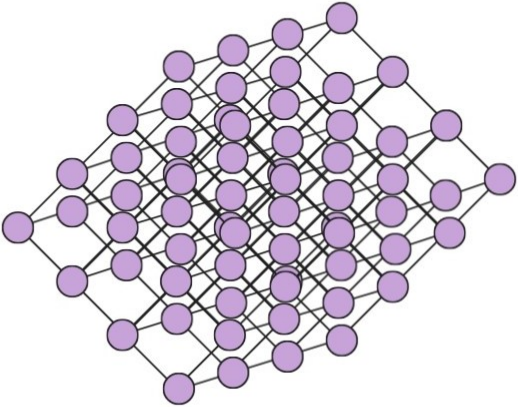
Given a poset $\mathcal{P}$

Given a poset $\mathcal{P}$

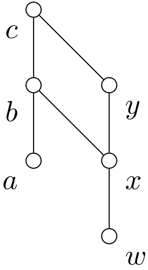


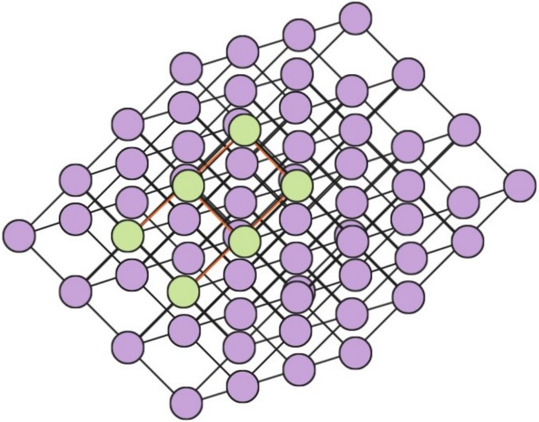
Given a poset $\mathcal{P}$



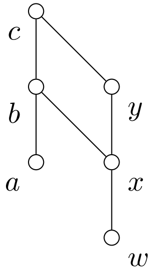


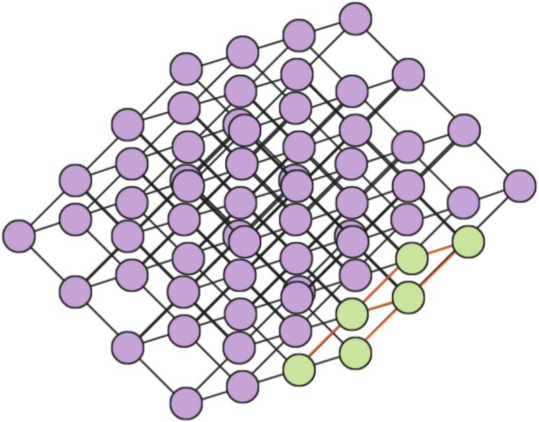
Given a poset $\mathcal{P}$



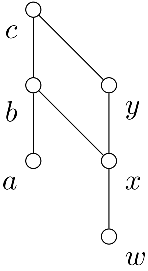


Given a poset $\mathcal{P}$

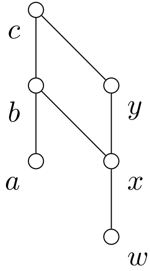


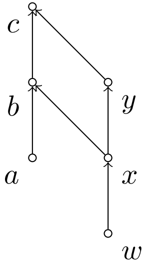


Given a poset $\mathcal{P}$

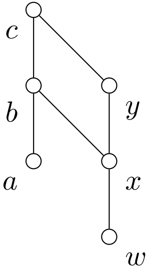


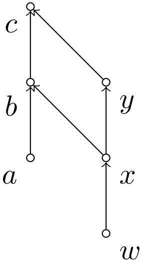
Given a poset $\mathcal{P}$

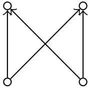




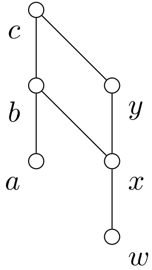
Given a poset $\mathcal{P}$

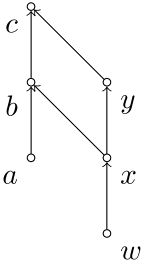


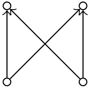


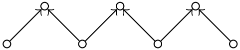


Given a poset $\mathcal{P}$









Krull-Schmidt category

Consider a Krull-Schmidt category $\mathcal{B}$ with the following four properties:

$A,1$ $\mathcal{B}$ has an exact structure with enough projectives and injectives.

$A,2$ $\mathcal{B}$ has almost split sequences.

$A,3$ There is an indecomposable projective object $\hat{S} \in \mathcal{B}$ such that $\mathrm{Hom}(\hat{S}, X) \neq 0$ for all $X \in \mathcal{B}$ and if $f : X \to \hat{S}$ is a non-zero morphism, then f is a retraction.

$A,4$ If $X \to Q$ and $Y \to Q$ are irreducible morphisms in $\mathcal{B}$, with Q indecomposable projective and X, Y indecomposable objects of $\mathcal{B}$, then $X \cong Y$.

Theorem

Let $\mathcal{B}$ be a category satisfying the previous conditions and let $\mathcal{C}$ be the Auslander-Reiten component of $\hat{S}$, the object of $\mathcal{B}$ in A,3. Then, there exists a "unique" set of sections $\{\mathcal{S}_i\}_{i\in I}$ in $\mathcal{C}$, where I is either the set of natural numbers or $I = \{1, 2, \ldots, n\}$, with the following properties

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.

(2) If $X \in \mathcal{S}_i$ and X is not injective then $\tau^{-1}X \in \mathcal{S}_{i+1}$.

(3) If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$.

(4) If $i \neq j$, then $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$.

(5) $\mathcal{C} = \bigcup_{i\in I} \mathcal{S}_i$.

Pseudo hereditary projectives

Let $\mathcal{A}$ be a Krull-Schmidt category with an exact structure having enough projectives and enough injectives. We call an indecomposable projective objet $M \in \mathcal{A}$, *pseudo hereditary projective* if for any chain of irreducible morphisms in $\mathcal{A}$:

$$X_1 \to X_2 \to \cdots \to X_l \to M$$

the objects $X_1, ..., X_l$ are projective in $\mathcal{A}$.

Defining $\{\mathcal{S}_\ell\}_{\ell \in I}$

$\mathcal{S}_1$ pseudo hereditary projectives.

Defining $\{\mathcal{S}_\ell\}_{\ell \in I}$

$\mathcal{S}_1$ pseudo hereditary projectives.

If $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ have been defined and all the objects in $\mathcal{S}_\ell$ are injectives, we set $I = \{1, 2, \ldots, \ell\}$.

Defining $\{\mathcal{S}_\ell\}_{\ell \in I}$

$\mathcal{S}_1$ pseudo hereditary projectives.

If $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ have been defined and all the objects in $\mathcal{S}_\ell$ are injectives, we set $I = \{1, 2, \ldots, \ell\}$.

Otherwise $\quad \mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell; \quad$ where

Defining $\{\mathcal{S}_\ell\}_{\ell \in I}$

$\mathcal{S}_1$ pseudo hereditary projectives.

If $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ have been defined and all the objects in $\mathcal{S}_\ell$ are injectives, we set $I = \{1, 2, \ldots, \ell\}$.

Otherwise $\quad \mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell; \quad$ where
$\underline{S}_\ell = \{Y \in \mathcal{C} | \tau Y \in \mathcal{S}_\ell\}$ and

Defining $\{\mathcal{S}_\ell\}_{\ell\in I}$

$\mathcal{S}_1$ pseudo hereditary projectives.

If $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ have been defined and all the objects in $\mathcal{S}_\ell$ are injectives, we set $I = \{1, 2, \ldots, \ell\}$.

Otherwise $\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell$; where
$\underline{S}_\ell = \{Y \in \mathcal{C} | \tau Y \in \mathcal{S}_\ell\}$ and
$\mathcal{T}_\ell$ is the set of projective indecomposable objects Z, for which there is a chain of irreducible morphisms $X \to Z_1 \to \cdots \to Z_t = Z$ with $Z_1, \ldots, Z_{t-1}$ projectives, $t \in \mathbb{N}$ and $X \in \underline{\mathcal{S}}_\ell$.

$\mathcal{S}_1$ is a section

$\mathcal{S}_1$ is a section

$X \to Y$ irreducible morphism with $X \in \mathcal{S}_1$ and $Y \in \mathcal{C}$.

$\mathcal{S}_1$ is a section

$X \to Y$ irreducible morphism with $X \in \mathcal{S}_1$ and $Y \in \mathcal{C}$.
Y is projective

$\mathcal{S}_1$ is a section

$X \to Y$ irreducible morphism with $X \in \mathcal{S}_1$ and $Y \in \mathcal{C}$.
Y is projective

Y is not projective

$\mathcal{S}_1$ satisfy the conditions (1) to (4)

$\mathcal{S}_1$ satisfy the conditions (1) to (4)

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.

$\mathcal{S}_1$ satisfy the conditions (1) to (4)

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.
(2) If $X \in \mathcal{S}_i$ and X is not injective, then $\tau^{-1} X \in \mathcal{S}_{i+1}$.

$\mathcal{S}_1$ satisfy the conditions (1) to (4)

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.
(2) If $X \in \mathcal{S}_i$ and X is not injective, then $\tau^{-1} X \in \mathcal{S}_{i+1}$.
(3) If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$.

$\mathcal{S}_1$ satisfy the conditions (1) to (4)

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.
(2) If $X \in \mathcal{S}_i$ and X is not injective, then $\tau^{-1} X \in \mathcal{S}_{i+1}$.
(3) If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$.
(4) If $i \neq j$, then $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$.

Induction

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

$$\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell.$$

Induction

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

$$\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell.$$

$$\mathcal{S}_j \cap \mathcal{S}_{\ell+1} = \emptyset \text{ for } j < \ell + 1.$$

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

$$\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell.$$

$$\mathcal{S}_j \cap \mathcal{S}_{\ell+1} = \emptyset \text{ for } j < \ell + 1.$$

$X \in \mathcal{S}_j \cap \mathcal{S}_{\ell+1}$,

Induction

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

$$\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell.$$

$$\mathcal{S}_j \cap \mathcal{S}_{\ell+1} = \emptyset \text{ for } j < \ell + 1.$$

$X \in \mathcal{S}_j \cap \mathcal{S}_{\ell+1}$, if X is not projective

Induction

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

$$\mathcal{S}_{\ell+1} = \underline{S}_\ell \cup \mathcal{T}_\ell.$$

$$\mathcal{S}_j \cap \mathcal{S}_{\ell+1} = \emptyset \text{ for } j < \ell + 1.$$

$X \in \mathcal{S}_j \cap \mathcal{S}_{\ell+1}$, if X is not projective

X projective

If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$

If $X \in \mathcal{S}_i$ and X is not injective, then $\tau^{-1}X \in \mathcal{S}_{i+1}$

If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$

If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$

There is a chain of irreducible morphisms
$Z \to Z_1 \to \cdots \to Z_t \to Y$ with $Z \in \underline{\mathcal{S}}_\ell$ and $Z_1, \ldots, Z_t$ projectives.

If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$

There is a chain of irreducible morphisms $Z \to Z_1 \to \cdots \to Z_t \to Y$ with $Z \in \underline{\mathcal{S}}_\ell$ and $Z_1, \ldots, Z_t$ projectives.

$$X \cong Z_t;$$

If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$

There is a chain of irreducible morphisms
$Z \to Z_1 \to \cdots \to Z_t \to Y$ with $Z \in \underline{\mathcal{S}}_\ell$ and $Z_1, \ldots, Z_t$ projectives.

$$X \cong Z_t;$$

$$X \in \mathcal{T}_\ell \subset \mathcal{S}_{\ell+1}.$$

$$\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$$

$\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$

If $Y \in \bigcup_{i \in I} \mathcal{S}_i$ and $W \to Y$ or $Y \to W$ are irreducible morphisms with $W \in \mathcal{C}$, then $W \in \bigcup_{i \in I} \mathcal{S}_i$.

$\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$

If $Y \in \bigcup_{i \in I} \mathcal{S}_i$ and $W \to Y$ or $Y \to W$ are irreducible morphisms with $W \in \mathcal{C}$, then $W \in \bigcup_{i \in I} \mathcal{S}_i$.

$$Y \to W$$

$\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$

If $Y \in \bigcup_{i \in I} \mathcal{S}_i$ and $W \to Y$ or $Y \to W$ are irreducible morphisms with $W \in \mathcal{C}$, then $W \in \bigcup_{i \in I} \mathcal{S}_i$.

$$Y \to W$$

either $W \in \mathcal{S}_i$ or W is not projective and $\tau W \in \mathcal{S}_i$

$\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$

If $Y \in \bigcup_{i \in I} \mathcal{S}_i$ and $W \to Y$ or $Y \to W$ are irreducible morphisms with $W \in \mathcal{C}$, then $W \in \bigcup_{i \in I} \mathcal{S}_i$.

$$Y \to W$$

either $W \in \mathcal{S}_i$ or W is not projective and $\tau W \in \mathcal{S}_i$

$$W \to Y$$

Unicity

Suppose $\{\mathcal{S}'_i\}_{i\in J}$ is a family of sections in $\mathcal{C}$ with the conditions $(1), (2), (3), (4)$.

Thank you

Dziękuję Ci