Universidad Nacional de Colombia - Bogotá

Preprojective component in a suitable Krull-Schmidt category

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Krull-Schmidt category

Consider a Krull-Schmidt category $\mathcal{B}$ with the following four properties:

$A,1$ $\mathcal{B}$ has an exact structure with enough projectives and injectives.

$A,2$ $\mathcal{B}$ has almost split sequences.

$A,3$ There is an indecomposable projective object $\hat{S} \in \mathcal{B}$ such that $\mathrm{Hom}(\hat{S}, X) \neq 0$ for all $X \in \mathcal{B}$ and if $f : X \to \hat{S}$ is a non-zero morphism, then f is a retraction.

$A,4$ If $X \to Q$ and $Y \to Q$ are irreducible morphisms in $\mathcal{B}$, with Q indecomposable projective and X, Y indecomposable objects of $\mathcal{B}$, then $X \cong Y$.

Theorem

Let $\mathcal{B}$ be a category satisfying the previous conditions and let $\mathcal{C}$ be the Auslander-Reiten component of $\hat{S}$, the object of $\mathcal{B}$ in A,3. Then, there exists a "unique" set of sections $\{\mathcal{S}_i\}_{i\in I}$ in $\mathcal{C}$, where I is either the set of natural numbers or $I = \{1, 2, \ldots, n\}$, with the following properties

(1) If $X \in \mathcal{S}_i$ and X is not projective, then $i > 1$ and $\tau X \in \mathcal{S}_{i-1}$.

(2) If $X \in \mathcal{S}_i$ and X is not injective then $\tau^{-1} X \in \mathcal{S}_{i+1}$.

(3) If $X \to Y$ is an irreducible morphism with $Y \in \mathcal{S}_i$ projective, then $X \in \mathcal{S}_i$.

(4) If $i \neq j$, then $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$.

(5) $\mathcal{C} = \bigcup_{i\in I} \mathcal{S}_i$.

Pseudo hereditary projectives

Let $\mathcal{A}$ be a Krull-Schmidt category with an exact structure having enough projectives and enough injectives. We call an indecomposable projective objet $M \in \mathcal{A}$, *pseudo hereditary projective* if for any chain of irreducible morphisms in $\mathcal{A}$:

$$X_1 \to X_2 \to \cdots \to X_l \to M$$

the objects $X_1, ..., X_l$ are projective in $\mathcal{A}$.

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$\underline{S}_\ell = \{Y \in \mathcal{C} | \tau Y \in \mathcal{S}_\ell\}$ and
$\mathcal{T}_\ell$ is the set of projective indecomposable objects Z, for which there is a chain of irreducible morphisms $X \to Z_1 \to \cdots \to Z_t = Z$ with $Z_1, \ldots, Z_{t-1}$ projectives, $t \in \mathbb{N}$ and $X \in \underline{\mathcal{S}}_\ell$.

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Induction

Suppose the sections $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ are built holding the conditions (1) to (4).

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either $W \in \mathcal{S}_j$ or W is not projective and $\tau W \in \mathcal{S}_j$

$$Y \in \mathcal{S}_j; \quad W \to Y$$

Y is projective
Y is non-projective
$\tau Y \in \mathcal{S}_{i-1}$ and there is an irreducible morphism $\tau Y \to W$

Unicity

Suppose $\{\mathcal{S}'_i\}_{i\in J}$ is a family of sections in $\mathcal{C}$ with the conditions $(1), (2), (3), (4)$.

Thank you