On τ-rigid modules over string algebras

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Motivation

- String algebra, Bulter-Ringel
- Geometric model
  - Assem-Brüstle-Charbonneau-Jodoi-Plamondon,
  - Baur-Simões,
  - He-Zhou-Zhu,
  - Opper-Plamondon-Schroll
  - ...
- τ-tilting theory, Adach-Iyama-Reiten.

- K: an algebraically closed field;
- A: a finite dimensional K-algebra;
- $|A|$: the rank of A;
- $\operatorname{mod} A$: category of finitely generated left A-modules;
- $\operatorname{proj} A$: subcategory of $\operatorname{mod} A$ whose objects are projective modules;
- τ: Auslander-Reiten translation;
- α^{-1}: the formal inverse of the arrow α such that $s(\alpha^{-1}) = t(\alpha), t(\alpha^{-1}) = s(\alpha)$;
- Q_1^{-1}: the set of formal inverse of Q_1.

String algebras

Definition

A finite dimensional K-algebra $A = KQ/I$ is called a string algebra, if it satisfies the followings:

(S1) For any vertex $a \in Q_0$, there are at most two incoming arrows and at most two outgoing arrows.

(S2) For any $\alpha \in Q_1$, there is at most one arrow β, such that $\alpha\beta \notin I$, and there is at most one arrow γ, such that $\gamma\alpha \notin I$.

(S3) I is generated by paths of length at least two.

Gentle algebras

Definition

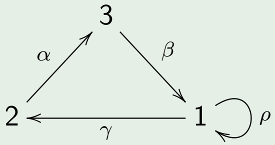
A string algebra is called a gentle algebra, if satisfies $S(1), S(2)$ and the followings:

(G1) for any $\alpha \in Q_1$, there is at most one $\theta \in Q_1$ such that $\alpha\theta \in I$, and at least one $\delta \in Q_1$, such that $\delta\alpha \in I$.

(G2) I is exactly generated by paths of length of 2.

Example

$Q =$



where $I = \{\rho^2, \alpha\beta, \beta\gamma, \gamma\alpha\}$.

String

Definition

A path $S = S_1 \dots S_m$, where $S_i \in Q_1 \cup Q_1^{-1}, 1 \leq i \leq m$ of length m is called a string if:

- $t(S_i) = s(S_{i+1})$
- neither subpath $S_i S_{i+1}$ nor its inverse belongs to I;
- $S_{i+1} \neq S_i^{-1}, 1 \leq i < m$.

Indecomposable modules over string algebras

Proposition(Butler- Ringel[1],1987)

1. Any indecomposable module over a string algebra is either string module or band module.
2. Band module is not τ-rigid.

Remark

When considering the τ-rigid modules over a string algebra, we only study the string modules.

[1] Butler-Ringel "Auslander-reiten sequences with few middle terms and applications to string algebras". Communications in Algebra

String modules

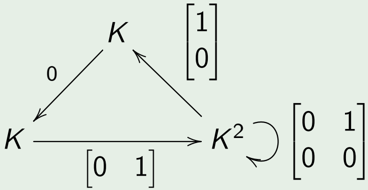
For a string $S = S_1 \dots S_m$, we define a representation $M(S)$ of Q as follows:

- Let $u(i) = s(S_{i+1}), 0 \leq i < m$, and $u(m) = t(S_m)$.
- Given a vertex a of Q, let $M(S)_a$ be the vector space with basis given by $\{x_i | u(i) = a\}$.
- if $S_i \in Q_1$, then define $S_i(x_{i-1}) = x_i$, if $S_i \in Q_1^{-1}$, then define $S_i(x_i) = x_{i-1}$.

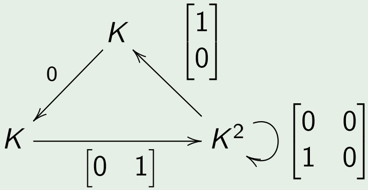
Example

Let $\lambda_1 = \alpha\rho\beta, \lambda_2 = \alpha\rho^{-1}\beta$, then

$M(\lambda_1) =$



$M(\lambda_2) =$



Morphisms between string modules

Definition

A substring $\rho = S_i \cdots S_j$ of a string $S = S_1 \dots S_m$ is said to be the:

- a top substring of S, if $i = 1$ or $S_{i-1} \in Q_1^{-1}$ and $j = n$ or $S_{j+1} \in Q_1$. The set of top substrings of S is denoted by $\mathrm{Top}(S)$.
- a bottom substring of S, if $i = 1$ or $S_{i-1} \in Q_1$ and $j = n$ or $S_{j+1} \in Q_1^{-1}$. The set of bottom substrings of S is denoted by $\mathrm{Bot}(S)$.

Proposition(Butler- Ringel[2],1987)

Let S, S' be two strings for a string algebra A. Then $\dim_K \operatorname{Hom}_A(M(S), M(S')) = |\{(u, u') \in \operatorname{Top}(S) \times \operatorname{Bot}(S') | u^{\pm 1} = u'\}|$.

Example

$\dim \operatorname{Hom}(M(\lambda_1), M(\lambda_1)) = 1$, and $\dim \operatorname{Hom}(M(\lambda_2), M(\lambda_2)) = 2$.

[2] Butler-Ringel "Auslander-reiten sequences with few middle terms and applications to string algebras". Communications in Algebra

τ-rigid pair

Definition

- M is a τ-rigid module, if $\mathrm{Hom}_A(M, \tau M) = 0$;
- A pair (M, P) with $M \in \mathrm{mod}\, A$ and $P \in \mathrm{proj}\, A$ is a τ-rigid pair, if M is τ-rigid, and $\mathrm{Hom}_A(P, M) = 0$;
- A τ-rigid pair (M, P) is an almost complete τ-tilting pair if $|M| + |P| = |A| - 1$.
- A τ-rigid pair (M, P) is a τ-tilting pair if $|M| + |P| = |A|$.

τ-reachable property

Definition

- Let $(M, P), (N, Q)$ be two τ-rigid pairs, (M, P) is τ-reachable from (N, Q), if there exist a sequence of non-zero τ-rigid pairs $(M_1, P_1), \ldots, (M_k, P_k)$ such that $(M \oplus M_1, P \oplus P_1), \ldots, (M_k \oplus N, P_k \oplus Q)$ are τ-rigid pairs.
- The algebra A has τ-reachable property if any two indecomposable τ-rigid pairs are τ-rechable from each other.

Auslander-Reiten translation of string modules

Definition

We say a string S:

- starts (resp.ends) in a deep if there is no arrow α such that $\alpha^{-1}S$ (resp.$S\alpha$) is a string.
- starts (resp.ends) in a peak if there is no arrow α such that αS (resp.$S\alpha^{-1}$) is a string.

Definition

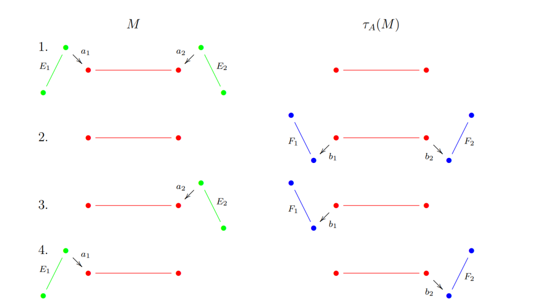
- Let S be a string that does not start in a deep and $\alpha, \alpha_1 \ldots, \alpha_m$ be arrows such that ${}_cS = \alpha_m \ldots \alpha_1 \alpha^{-1} S$ starts on a peak. We say that ${}_cS$ is obtained by adding a cohook at the start of S.
- Let S be a string that start in a deep and $\alpha, \alpha_1 \ldots, \alpha_m$ be arrows then there is a string ${}_{h^{-1}}S = S'$, such that $S = \alpha_m^{-1} \ldots \alpha_1^{-1} \alpha S'$. We say that S' is obtained by removing a hook at the start of S.

For a string S, we define $\tau M(S) = M(S')$, where S' is obtained by the translation in the following rules:

- if S does not start (resp.end) in a deep, then add a cohook at the start (resp.end) of S.
- if S start (resp.end) in a deep, then remove a hook at the start (resp.end) of S.

Remark

The following figure helps us understand the Auslander-Reiten translation of the string modules:



Main results

Theorem

The indecomposable τ-rigid over a finite dimensional string algebra is completely determined by the endpoints and the arrows of its corresponding string.

Theorem

Let A be a finite dimensional gentle algebra with $|A| > 1$, then A has τ-reachable property.

Remark

The proof does not rely on the geometric model for the module categories of (skew-)gentle algebras.

Thank you!