On HH^1 and the fundamental groups

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joint with Benjamin Briggs

The fundamental group

Setup: Let Q be a finite quiver and let k be an algebraically closed field k of characteristic zero. Let kQ be the path algebra of Q. Let A be is a finite dimensional basic algebra over k . Let $\nu : kQ \to A$ be a presentation. Let $I := \mathrm{Ker}(\nu) \subseteq kQ$ be an admissible ideal, and $A \cong kQ/I$.

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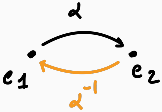

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Homotopy relation: Equivalence relation generated by:

- $\alpha^{-1}\alpha \sim s(\alpha)$ and $\alpha\alpha^{-1} \sim t(\alpha)$ for any arrow α
- if $v \sim v'$, then $uvw \sim uv'w$, where u, v, v', w walks.
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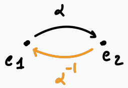
Example. Fix $e = e_1$. Then $\pi_1(Q; 0, e_1) = \{e_1\}$.



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If I is a monomial ideal in kQ then $\pi_1(Q; I) = \pi_1(Q; (0))$ is the free group on the number of holes (= $|Q_1| - |Q_0| + |\text{conn. components}|$) of Q as a graph.

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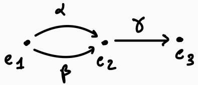

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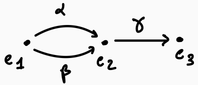

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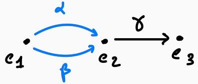

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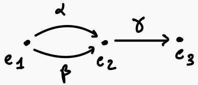

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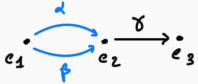

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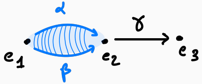

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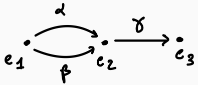

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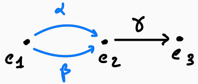

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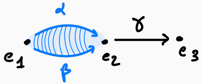

But $kQ/I \cong kQ/J$...

Problem. We want invariants, but the fundamental group depends on a presentation!

Spoiler. Solution is to consider all the presentations and fundamental groups

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The first degree Hochschild cohomology is a Lie algebra. The Lie bracket in $\mathrm{HH}^1(A)$ is defined as:

$$[f, g] := f \circ g - g \circ f.$$

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In particular, $\mathrm{mt} - \mathrm{rank}(\mathrm{HH}^1(A))$ is a derived invariant.

Main results

Theorem. (Assem-de la Peña, de la Peña-Saorín). For any presentation $A \cong kQ/I$ there is a canonical embedding

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Theorem. (Farkas, Green, Marcos) (Le Meur) (Briggs, RyD). For any finite dimensional basic k-algebra A, all maximal tori in $\mathrm{HH}^1(A)$ are of the form $\sigma(\mathrm{Hom}(\pi_1(Q;I);k))$ for some presentation $A \cong kQ/I$.

Main results and applications

Corollary.

$$\pi_1-\text{rank}(A) := \max\{\dim_k \text{Hom}(\pi_1(Q;I),k) \text{ over all presentations}\}$$
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This was proven for gentle algebras by Avella-Alaminos and Geiss.

THANK YOU!