Derived equivalences of algebras VS equivalence relations of matrices

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Goal: To introduce
new equivalence relations on matrices
for understanding

- **Morita equivalences**
- **derived equivalences**
- **stable equivalences of Morita type**

among centralizer algebras of matrices

Joint work with Xiaogang Li

- [X.G. Li + X.: arXiv:2312.08794]

1. Introduction
   - **Equivalence relations on matrices**
     - Known equivalence relations
     - New equivalence relations
   - **Equivalences of algebras**
     - Morita and derived equivalences
2. Main results
   - **Centralizers of matrices**
     - Definitions and Examples
   - **Main result**
     - Theorem on Morita and derived equivalences
     - Example and problems

Definitions and notation

R :	field
$R[x]$:	poly. alg./R in variable x
A :	Associative algebra/R with 1
$M_{n\times m}(A)$:	$n \times m$ matrix ring over A
$M_n(A)$:	$n \times n$ matrix ring over A
e_{ij} :	Matrix units
$GL_n(A)$:	General lin. group over A of deg. n
Σ_n	Symmetric group of permutations on$\{1, 2, \cdots, n\}$

Definitions and notation

A-mod:	Category of all f. g. left A-modules
add(M):	Additive cat. gen. by $M \in A$-mod
A-proj:	Subcat. of f. g. proj. left A-mod.s
$\mathscr{K}^{\rm b}(A\text{-proj})$:	Homotopy cat. of f.g. proj. A-mod.s
$\mathscr{D}^b(A)$:	(bounded) Derived category of A-mod

- **Known equ. rel.s of matrices**
  - Transform. equivalence: $a, b \in M_{n\times m}(R)$,

$$a \sim^T b \Longleftrightarrow \exists\, p \in GL_n(R), q \in GL_m(R), b = paq.$$

  - Congruence equivalence: $a, b \in M_n(R)$,

$$a \sim^C b \Longleftrightarrow \exists\, p \in GL_n(R), b = pap^T.$$

  - Similarity equivalence: $a, b \in M_n(R)$,

$$a \sim^S b \Longleftrightarrow \exists\, p \in GL_n(R), b = pap^{-1}.$$

• New equivalence relations

$c \in M_n(R)$

◇ Elementary divisors of c:

- $xI - c \in M_n(R[x])$ is transformed into **Smith Normal Form**

$$\begin{pmatrix} d_1(x) & & & & \\ & d_2(x) & & & \\ & & \ddots & & \\ & & & d_r(x) & \\ & & & & 0_{(n-r)\times(n-r)} \end{pmatrix}, d_i(x) \mid d_{i+1}(x)$$

- $d_1(x), \cdots, d_s(x)$: positive degree, decompose each of them into product of powers of irred. poly.s.
- Elementary divisors: All of these powers of irred. poly.s (counting multiples)

Notation

♢ $\mathcal{E}_c \subset R[x]$: the set of **elementary divisors** of c

♢ $\mathcal{M}_c$: the set of **maximal divisors** of c, defined

$$\{ f(x) \in \mathcal{E}_c \mid f(x) \text{ is max. w.r.t. division order } \}$$

♢ **Division order:**

$f(x), g(x) \in R[x]$, positive degree, define

$$f(x) \leq g(x) \overset{def}{\Longleftrightarrow} f(x) \mid g(x)$$

Notation

$\diamondsuit$ For $f(x) \in \mathcal{M}_c$, the set of **power indices** of $f(x)$:

$$P_c(f(x)) := \{t \geq 1 \mid \exists \text{ irrd. factor } p(x) \text{ of } f(x) \text{ s.t. } p(x)^t \in \mathcal{E}_c\},$$

$\diamondsuit$ For $T := \{m_1, m_2, \cdots, m_s\}$ of positive integers, $m_1 > m_2 > \cdots > m_s$, define multisets

$$\mathcal{H}_T := \{m_1 - m_2, \cdots, m_{s-1} - m_s, m_s\}$$

$c \in M_n(R), d \in M_m(R)$

Definition (M-equivalence)

- $c \overset{M}{\sim} d$: $\exists$ *bijection* $\pi : \mathcal{M}_c \to \mathcal{M}_d$ *s. t.*
  - $\diamond$ $R[x]/(f(x)) \simeq R[x]/(\pi(f(x)))$ *as alg.s*
  - $\diamond$ $P_c(f(x)) = P_d(\pi(f(x)))$ *for all* $f(x) \in \mathcal{M}_c$.

$\star$ $c \overset{M}{\sim} d$: equiv. relation on all square matrices / R

Definition (D-equivalence)

- $c \overset{D}{\sim} d$ *if* $\exists$ *bijection* $\pi : \mathcal{M}_c \to \mathcal{M}_d$ *s. t.*
  - $\diamond$ $R[x]/(f(x)) \simeq R[x]/(\pi(f(x)))$ *as alg.s*
  - $\diamond$ $\mathcal{H}_{P_c(f(x))} = \mathcal{H}_{P_d(\pi(f(x)))}$ *for all* $f(x) \in \mathcal{M}_c$.

$\star$ $c \overset{D}{\sim} d$: equiv. relation

Example 1

$J_n(\lambda)$: Jordan block with eigenvalue $\lambda \in R$

(1) $c = J_3(1) \oplus J_4(1) \oplus J_3(0) \oplus J_2(0)$

$d = J_3(0) \oplus J_4(0) \oplus J_3(1) \oplus J_2(1)$

- $\mathcal{E}_c = \{x^2, x^3, (x-1)^3, (x-1)^4\}$,
  $\mathcal{M}_c = \{x^3, (x-1)^4\}$,
  $P_c(x^3) = \{2, 3\}, P_c((x-1)^4) = \{3, 4\}$,
- $\mathcal{E}_d = \{x^3, x^4, (x-1)^2, (x-1)^3\}$,
  $\mathcal{M}_d = \{x^4, (x-1)^3\}$,
  $P_d(x^4) = \{3, 4\}$, $P_d((x-1)^3) = \{2, 3\}$.

Example 1

- $\pi : \mathcal{M}_c \to \mathcal{M}_d$, $x^3 \mapsto (x-1)^3$, $(x-1)^4 \mapsto x^4$.

$$c \overset{M}{\sim} d$$

(2) $a = J_5(0) \oplus J_4(0) \oplus J_2(0)$

$b = J_5(0) \oplus J_3(0) \oplus J_1(0)$

$\Longrightarrow$

$$a \overset{D}{\sim} b, \quad a \overset{M}{\not\sim} b$$

Morita and derived equivalences for algebras

Definition

A, B: *f. d. algebras/field*

$A \overset{der}{\sim} B$: *derived equivalent if* $\mathscr{D}^b(A)$ *and* $\mathscr{D}^b(B)$ *are equiv. as triangulated cat.s.*

Special case:

Definition

$A \overset{Morita}{\sim} B$: *Morita equivalent if* A-mod *and* B-mod *are equivalent.*

Example 2: der. equiv. algebras

R: ring, I : ideal of R

$$\begin{pmatrix} R & I & \cdots & I \\ R & \ddots & \ddots & \vdots \\ \vdots & \ddots & R & I \\ R & \cdots & R & R \end{pmatrix}_n \overset{der}{\sim} \begin{pmatrix} R & R/I & \cdots & R/I \\ 0 & R/I & \ddots & \vdots \\ \vdots & \ddots & \ddots & R/I \\ 0 & \cdots & 0 & R/I \end{pmatrix}_n$$

- [X.: Math. Z. 273 (2013) 1025-1052]

Special class of algebras

Definition

$C \subseteq M_n(R)$

- *Centralizer algebra of C in $M_n(R)$:*

$S_n(C,R) := \{a \in M_n(R) \mid ac = ca \ \forall c \in C\}$

$$S_n(C,R) = \bigcap_{c \in C} S_n(c,R)$$

where $S_n(c,R) := S_n(\{c\},R)$

- $S_n(c,R)$ are called *centralizer matrix algebra*

Frobenius's Theorem

Theorem (F. G. Frobenius, 1849-1917)

- R: *field*, $c \in M_n(R)$
- $d_1(x), \cdots, d_s(x)$: *all invariant factors of* c *with*
- $n_i := \deg(d_i) \geq 1$, $1 \leq i \leq s$

$\Longrightarrow$

$$\dim_R(S_n(c,R)) = \sum_{j=1}^{s}(2s-2j+1)n_j$$

- [N.Jacobson, Basic algebra I, p.200.]

Example 3

$c := \text{diag}(J_s, J_{s-1}, \ldots, J_1) \in M_n(R)$: Jordan-form matrix, $r \in R$

$$J_i = \begin{pmatrix} r & 1 & \cdots & 0 & 0 \\ 0 & r & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & r & 1 \\ 0 & 0 & \cdots & 0 & r \end{pmatrix}_{i \times i}$$

Example 3 (continued)

$S_n(c, R)$: quasi-hereditary alg.:

$$\underset{1}{\bullet} \underset{\alpha_1}{\overset{\beta_1}{\rightleftarrows}} \underset{2}{\bullet} \underset{\alpha_2}{\overset{\beta_2}{\rightleftarrows}} \underset{3}{\bullet} \cdots \bullet \underset{\alpha_{s-2}}{\overset{\beta_{s-2}}{\rightleftarrows}} \bullet \underset{\alpha_{s-1}}{\overset{\beta_{s-1}}{\rightleftarrows}} \underset{s}{\bullet}$$

$$\beta_{s-1}\alpha_{s-1} = 0, \ \alpha_i\beta_i = \beta_{i-1}\alpha_{i-1}, 1 < i < s.$$

- Auslander algebra of $R[X]/(X^s)$.
- [X. + J. B. Zhang, Linear Alg. Appl. 622 (2021) 215-249]

- Modular Hecke algebras: modular rep. theory of groups [J. Alperin, 1989]
- Nilpotent matrices: semisimple Lie algebras [A. Premet, Invent. Math. 154(2003)]
- Maximal doubly stochastic matrices [Cruz-Dolinar-Fernandes-Kuzma, 2017]
- Structural aspects: Cellular algebra$/k = \bar{k}$, Sep. and Frobenius extensions [Our group, 2019-2022].

Our questions on centralizer matrix algebras

$$c \in M_n(R), \; d \in M_m(R)$$

- $S_n(c, R) \overset{Morita}{\sim} S_m(d, R)$?
- $S_n(c, R) \overset{der}{\sim} S_m(d, R)$?

Main results: Derived equivalences

Theorem (A)

R: field, $c \in M_n(R), d \in M_m(R)$

- $S_n(c, R) \overset{Morita}{\sim} S_m(d, R) \Leftrightarrow c \overset{M}{\sim} d$
- $S_n(c, R) \overset{der}{\sim} S_m(d, R) \Longleftrightarrow c \overset{D}{\sim} d$

- [X.G. Li + X. : Derived and stable equivalences of centralizer algebras. arXiv:2312.08794]
- Stable equivalences of Morita type were considered

Special matrices: permutation matrices

$\sigma \in \Sigma_n$,

- $\sigma = \sigma_1 \cdots \sigma_s$: dis. cycle-perm.s σ_i
- Its cycle type: $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\lambda_i \geq 1$,
- Permutation matrix of σ:

$$c_\sigma := \sum_{i=1}^{n} e_{i,\sigma(i)}$$

$p > 0$: prime,

Definition

- σ_i: p-*regular* if $p \nmid \lambda_i$, and p-*singular* if $p \mid \lambda_i$.
- $r(\sigma)$: *product of all* p*-regular cycles of* σ,
- $s(\sigma)$: *product of all* p*-singular cycles of* σ,

- If $p = 0$, all cycles are p-regular.
- $r(\sigma)$, $s(\sigma)$ as elements in Σ_n.

Proposition

$\textit{char}(R) = p \geq 0$, $\sigma \in \Sigma_n$, $\tau \in \Sigma_m$

$S_n(c_\sigma, R) \overset{der}{\sim} S_m(c_\tau, R) \Longrightarrow$

(1) $S_n(c_{r(\sigma)}, R) \overset{der}{\sim} S_m(c_{r(\tau)}, R)$

(2) $S_n(c_{s(\sigma)}, R) \overset{der}{\sim} S_m(c_{s(\tau)}, R)$

How about the converse? No! See arXiv:2312.08794.

Proposition

$\textit{char}(R) = p \geq 0$, $\sigma \in \Sigma_n$, $\tau \in \Sigma_m$

$S_n(c_\sigma, R) \overset{der}{\sim} S_m(c_\tau, R) \Longrightarrow$

(1) $S_n(c_{r(\sigma)}, R) \overset{der}{\sim} S_m(c_{r(\tau)}, R)$

(2) $S_n(c_{s(\sigma)}, R) \overset{der}{\sim} S_m(c_{s(\tau)}, R)$

How about the converse? No! See arXiv:2312.08794.

Example 1 (continued)

- $a = J_5(0) \oplus J_4(0) \oplus J_2(0)$

  $b = J_5(0) \oplus J_3(0) \oplus J_1(0)$

In Example 1(2), $a \overset{D}{\sim} b$, but $a \overset{M}{\not\sim} b$

- $S_{11}(a, R) \overset{der}{\sim} S_9(b, R)$
- $S_{11}(a, R) \overset{Morita}{\not\sim} S_9(b, R)$

R: field, $c \in M_n(R), d \in M_m(R)$

(1) Describe necessary and sufficient conditions for

$$S_n(c, R) \overset{st}{\sim} S_m(d, R)$$

(2) Describe **necessary and sufficient** conditions for

$$S_n(c, R) \overset{st.M}{\sim} S_m(d, R)$$

- Result for R: perfect field or condition on matrices c, d

Problems

(3) **Generalize the main result to comm. subalgebras?**

Given comm. R-subalg.s $C \subseteq M_n(R)$, $D \subseteq M_m(R)$, find neces. and suff. cond.s on C, D s.t. $S_n(C,R) \overset{der}{\sim} S_m(D,R)$

(4) **Describe equivalences of the singularity categories** of algebras $S_n(c,R)$?

Problems

(5) **What is the canonical form** of $c \in M_n(R)$ under $\overset{D}{\sim}$ or $\overset{M}{\sim}$? (for n minimal?)

(6) **Generalize all results to** R **being PID.**

Derived equivalences of algebras
VS
equivalence relations of matrices

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Thank you !

for your attention

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[Xiaogang Li + Changchang Xi: arXiv:2312.08794]