The first Hochschild cohomology groups under gluings

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Outline

- The stable equivalences by gluing
- The Hochschild cohomology groups
- Main results
    - Gluing vertices
    - Gluing arrows

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The notation

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- (Q, I) the bound quiver with Q finite and I an admissible ideal
- $\mathscr{B} := Q_{\geq 0}$ the set of all finite paths in Q, k-basis of kQ
- A, B finite dimensional quiver k-algebras of the form kQ/I

Stable equivalence by gluing

Theorem (Martinez-Villa 1979; Koenig, Liu 2008)

Let A and B be finite dimensional k-algebras such that B is a radical embedding of A. Then the following are equivalent:

(1) *A is stably equivalent to B;*

(2) *B is obtained from A by a finite number of steps of gluing a source vertex and a sink vertex.*

- This is a class of stable equivalences not of Morita type, but still close to Morita type

[1] R. Martinez-Villa, *Algebras stably equivalent to ℓ-hereditary*. in: Lecture Notes in Math. **832** (1979), pp. 396-431.

[2] S. Koenig and Y. Liu, *Gluing of idempotents, radical embeddings and two classes of stable equivalences*. J. Algebra **319** (2008), 5144-5164.

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The Hochschild cohomology

Definition

The *nth Hochschild cohomology group* *of algebra* A*, denoted by* $HH^n(A)$*, is defined by* $HH^n(A) := Ext^n_{A^e}(A, A)$*, where* $A^e := A \otimes_k A^{op}$ *is the enveloping algebra of* A*.*

- $HH^n(A)$ can be computed using different projective resolutions of A over its enveloping algebra A^e.
- We often use the reduced bar resolution of A which is introduced by C. Cibils.

Hochschild cohomology of lower degrees

The Hochschild cohomology groups of lower degrees have nice interpretation.

Fact

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(1) *The 0th Hochschild cohomology* $HH^0(A)$ *is the center of* A.

(2) *The first Hochschild cohomology* $HH^1(A) \cong Der(A, A)/Inn(A, A)$*, where*

$$Der(A, A) = \{f \in Hom_k(A, A) | f(ab) = af(b) + f(a)b, \forall a, b \in A\}$$
$$Inn(A, A) = \{g_b \in Hom_k(A, A) | g_b(a) = ab - ba, \forall a, b \in A\}$$

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(3) *In particular, $HH^1(A)$ is a Lie algebra with bracket*

$$[f,g] = f \circ g - g \circ f.$$

[3] M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. Math. **78** (2) (1963), 267-288.

Invariant

If algebras A and B are derived equivalent, then

- their Hochschild cohomology groups are isomorphic: $HH^n(A) \cong HH^n(B), \forall n \geq 0$. (J. Rickard 1991)
- in particular, we have Lie algebra isomorphism:

$$HH^1(A) \cong HH^1(B)$$

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If A and B are self-injective algerbas and $A \overset{St.M}{\simeq} B$, then

- $HH^n(A) \cong HH^n(B), \forall n > 0$. (Z. Pogorzaly 2001)
- in particular, HH^1 of two symmetric algebras are isomorphic as Lie algebras. (Koenig, Liu, Zhou 2012)

Question

Let B be obtained from A by gluing two vertices in Q_A.

- If we glue a source and a sink, then $A \overset{St}{\simeq} B$ but not of Morita type. What is the relation between $HH^n(A)$ and $HH^n(B)$?

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In this talk, I will give the answers of the above questions when $n = 0, 1$ for any finite dimensional quiver algebras.

Gröbner basis

Before presenting the approach to compute HH^1, we recall the Gröbner basis theory and parallel paths for kQ/I.

- Fix an admissible well-order on $\mathcal{B}$, that is, a well order which is compatible with multiplication
- For each $r \in kQ$, its tip $\mathrm{Tip}(r)$ is the maximal monomial p appearing with nonzero coefficient $c_r(p)$ in r

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- Put $\mathrm{Tip}(X) := \{\mathrm{Tip}(r) \mid r \in X, r \neq 0\}$ for $\emptyset \neq X \subseteq kQ$
- A Gröbner basis of I w.r.t. the admissible order is a subset $\mathcal{G} \subseteq I$ such that $\mathrm{Tip}(\mathcal{G})$ and $\mathrm{Tip}(I)$ generate the same ideal

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- Two paths α, β of Q are called parallel if $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$, denoted by $\alpha \| \beta$
- If X, Y are sets of paths of Q, we denote by
$$X \| Y := \{\alpha \| \beta \mid \alpha \in X, \beta \in Y\}$$
- Let $k(X \| Y)$ be the k-vector space with basis $X \| Y$

Compute HH^1

Proposition (Bardzell 1997)

Let $A = kQ/I$ be a quiver algebra with $\mathcal{G}$ as a reduced Gröbner basis of I. Then there is a minimal projective resolution of A as an A-bimodule

$$A \otimes_E k(\mathrm{Tip}(\mathcal{G})) \otimes_E A \xrightarrow{d_1} A \otimes_E kQ_1 \otimes_E A \xrightarrow{d_0} A \otimes_E kQ_0 \otimes_E A \xrightarrow{\mu} A \to 0$$

where $E \simeq kQ_0$ is the separable subalgebra of A and the A-bimodule morphisms are given by

$$\mu(a \otimes_E e_i \otimes_E b) = ae_ib,$$
$$d_0(a \otimes_E \alpha \otimes_E b) = a\alpha \otimes_E b - a \otimes_E \alpha b \text{ and}$$

$$d_1(a \otimes_E \mathrm{Tip}(g) \otimes_E b) = \sum_p c_g(p) \sum_{i=1}^{n} a\alpha_n \cdots \alpha_{i+1} \otimes_E \alpha_i \otimes_E \alpha_{i-1} \cdots \alpha_1 b$$

for all $p = \alpha_n \cdots \alpha_1 \in \mathrm{Supp}(g)$; $a, b \in A$; $e_i \in Q_0$; $\alpha, \alpha_n, \cdots, \alpha_1 \in Q_1$ and $g \in \mathcal{G}$

Compute HH^1

Proposition (Strametz; Rubio y Degrassi, Schroll, Solotar; Liu, Xing)

Let $A \simeq kQ/I$ be a quiver algebra with $\mathcal{G}$ as a reduced Gröbner basis, then $HH^n(A)$ for $n = 0, 1$ can be computed by the complex

$$0 \to k(Q_0 \parallel \mathcal{B}) \xrightarrow{\delta^0} k(Q_1 \parallel \mathcal{B}) \xrightarrow{\delta^1} k(\mathrm{Tip}(\mathcal{G}) \parallel \mathcal{B}) \xrightarrow{\delta^2} \cdots$$

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where the differentials are given by:

$$\delta^0 : k(Q_0 \parallel \mathcal{B}) \to k(Q_1 \parallel \mathcal{B})$$

$$e \parallel \gamma \mapsto \sum_{s(a)=e, a\gamma \in \mathcal{B}} a \parallel a\gamma - \sum_{t(a)=e, \gamma a \in \mathcal{B}} a \parallel \gamma a,$$

$$\delta^1 : k(Q_1 \parallel \mathcal{B}) \to k(\mathrm{Tip}(\mathcal{G}) \parallel \mathcal{B})$$

$$a \parallel \gamma \mapsto \sum_{r \in \mathcal{G}, p \in \mathrm{Supp}(r), p^{a\parallel\gamma} \in \mathcal{B}} c_r(p) \ \mathrm{Tip}(r) \parallel p^{a\parallel\gamma}.$$

Compute HH^1

Theorem (S 2006; RSS 2022; LX 2023)

Let A be a finite dimensional quiver algebra. Then the bracket

$$[a\|\alpha, b\|\beta] = b\|\beta^{a\|\alpha} - a\|\alpha^{b\|\beta} \qquad (a\|\alpha, b\|\beta \in Q_1\|\mathcal{B})$$

induces a Lie algebra structure on $\mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$ such that there is a natural isomorphism as Lie algebras

$$\mathrm{HH}^1(A) \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$$

[4] C. Strametz, *The Lie algebra structure on the first Hochschild cohomology group of a monomial algebra*. J. Algebra Appl. **5** (3) (2006), 245-270.

[5] L. Rubio y Degrassi, S. Schroll, A. Solotar, *The first Hochschild cohomology as a Lie algebra*. Quaestiones Mathematicae, 2022.

[6] Y. Liu and B. Xing, *Generalized parallel paths method for computing the first Hochschild cohomology groups with applications to brauer graph algebras*. arXiv: 2306.14372v3 (2023).

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HH^1 under gluing

Theorem (Liu, Rubio y Degrassi, Wen 2022)

Let $A = kQ_A/I_A$ be a quiver algebra and let $B \simeq kQ_B/I_B$ be obtained from A by gluing a source vertex e_1 and a sink vertex e_n in Q_A. Then

HH^1 under gluing

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(1) *If e_1 and e_n are from different blocks of A, then there is a Lie algebra isomorphism $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$;*

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(2) *If e_1 and e_n are from the same block of A, then $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathfrak{J}$ as Lie algebras, where $\dim(\mathfrak{J}) = 1$;*

(3) *Moreover, if A is a monomial algebra and $\mathrm{char}(k) = 0$, then the one-dimensional Lie ideal $\mathfrak{J}$ lies in the center of $\mathrm{HH}^1(B)$ and*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times \mathfrak{J} \simeq \mathrm{HH}^1(A) \times k$$

as Lie algebras.

HH^1 under gluing

Let B be obtained from A by gluing two arbitrary vertices. Then

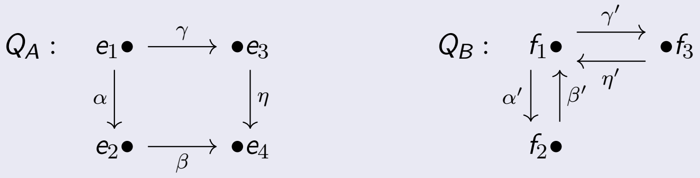
- we can provide explicit formula between dimensions of $HH^1(A)$ and $HH^1(B)$ in terms of some combinatorial datum.

[7] https://arxiv.org/abs/2211.05435

Example

Example

$B \simeq kQ_B/I_B$ is obtained from $A = kQ_A/I_A$ by gluing a source vertex e_1 and a sink vertex e_4 in Q_A:



- $I_A = \langle \beta\alpha - \eta\gamma \rangle$, after gluing the new relations are given by $Z_{new} = \{\alpha'\beta', \gamma'\beta', \alpha'\eta', \gamma'\eta'\}$, hence $I_B = \langle I_A \cup Z_{new} \rangle$.
- Fix an order on $(Q_A)_1$ by $\eta \succ \gamma \succ \beta \succ \alpha$, it follows that $\mathcal{G}_A = \{\eta\gamma - \beta\alpha\}$ and $\mathcal{G}_B = \mathcal{G}_A \cup Z_{new}$.
- We have $HH^1(A) = 0$ and $HH^1(B) \simeq \langle \alpha' \parallel \alpha' + \gamma' \parallel \gamma' \rangle \simeq k$.

Example

$B \simeq kQ_B/I_B$ is obtained from $A = kQ_A/I_A$ by gluing e_1 and e_4 from different blocks of A:

$$Q_A: \quad e_2 \bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet e_4 \qquad \bullet e_1 \xrightarrow{\beta} \bullet e_3$$

$$Q_B: \quad f_2 \bullet \underset{\alpha_2'}{\overset{\alpha_1'}{\rightrightarrows}} \bullet f_1 \xrightarrow{\beta'} \bullet f_3$$

- $I_A = 0$, after gluing the new relations are given by $Z_{new} = \{\beta'\alpha_1', \beta'\alpha_2'\}$, hence $I_B = \langle \beta'\alpha_1', \beta'\alpha_2' \rangle$.
- Let $char(k) \neq 2$, we have $HH^1(A) \simeq sl_2(k) \simeq HH^1(B)$.

Example

$B \simeq kQ_B/I_B$ is obtained from $A = kQ_A/I_A$ by gluing e_1 and e_3 from the same block of A:

$$Q_A: \; e_1 \bullet \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} e_2 \bullet \xrightarrow{\beta} \bullet e_3 \qquad\qquad Q_B: \; f_1 \bullet \;\; \begin{matrix} \xrightarrow{\alpha_1'} \\ \xrightarrow{\alpha_2'} \\ \xleftarrow[\beta']{} \end{matrix} \;\; \bullet f_2$$

- $I_A = \langle \beta\alpha_1, \beta\alpha_2 \rangle$, after gluing the new relations are given by $Z_{new} = \{\alpha_1'\beta', \alpha_2'\beta'\}$, hence $I_B = \langle \beta'\alpha_1', \beta'\alpha_2', \alpha_1'\beta, \alpha_2'\beta' \rangle$.
- Let $char(k) \neq 2$, we have $HH^1(A) \simeq sl_2(k)$ and $HH^1(B) \simeq gl_2(k)$. Moreover, $HH^1(B) \simeq HH^1(A) \times k$.

HH^0 under gluing

Proposition (Liu, Rubio y Degrassi, Wen 2022)

Let A be a quiver algebra and B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then

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Let A be a quiver algebra and B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then

(1) *If e_1 and e_n are from the same block, then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$.*

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(1) *If e_1 and e_n are from the same block, then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$.*

(2) *If e_1 and e_n are from different blocks, then the radical embedding $B \to A$ restricts to a radical embedding $Z(B) \to Z(A)$. In particular, $\dim Z(A) = \dim Z(B) + 1$.*

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Gluing arrows

We are particularly interested in the case of gluing a source arrow and a sink arrow.

Definition

An arrow α in Q_A is called as a source arrow if the following two conditions are satisfied:

(1) *$s(\alpha)$ is a source vertex;*

(2) *except α, there is neither an arrow starting from $s(\alpha)$ nor an arrow ending at $t(\alpha)$.*

Dually, an arrow β in Q_A is called as a sink arrow if the following two conditions are satisfied:

(1) *$t(\beta)$ is a sink vertex;*

(2) *except β, there is neither an arrow starting from $s(\beta)$ nor an arrow ending at $t(\beta)$.*

Gluing arrows

Theorem (Liu, Rubio y Degrassi, Wen 2024)

Let $A = kQ_A/I_A$ be a monomial algebra and let $B \simeq kQ_B/I_B$ be obtained from A by gluing a source arrow $\alpha : e_1 \to e_2$ and a sink arrow $\beta : e_{n-1} \to e_n$ in Q_A. Then

(1) *If α and β are from different blocks of A, then there is a Lie algebra isomorphism $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$;*

(2) *If α and β are from the same block of A, then $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathfrak{J}$ as Lie algebras, where $\dim(\mathfrak{J}) = 1$;*

(3) *Moreover, if $\mathrm{char}(k) = 0$, then the one-dimensional Lie ideal $\mathfrak{J}$ lies in the center of $\mathrm{HH}^1(B)$ and*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times \mathfrak{J} \simeq \mathrm{HH}^1(A) \times k$$

as Lie algebras.

[8] Y. Liu, L. Rubio y Degrassi and C. Wen, *The Hochschild cohomology groups under gluing arrows*. Comm. Algebra **52** (9) (2024), 3871-3897.

Gluing arrows

Proposition (Liu, Rubio y Degrassi, Wen 2024)

Let A be a monomial algebra and B be obtained from A by gluing two arrows $\alpha : e_1 \to e_2$ and $\beta : e_{n-1} \to e_n$ in Q_A. Then

(1) *If α and β are from the same block, then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$.*

(2) *If α and β are from different blocks, then there is an algebra monomorphism $Z(B) \hookrightarrow Z(A)$. In particular,*

$$\dim Z(A) = \dim Z(B) + 1$$

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Example

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The algebra B is obtained from A by gluing a source arrow α *and a sink arrow* β*:*

$$Q_A:\ e_1\bullet \xrightarrow{\alpha} \bullet e_2 \xrightarrow{\eta} \bullet e_3 \xrightarrow{\beta} \bullet e_4 \qquad Q_B:\ f_1\bullet \underset{\eta^*}{\overset{\gamma^*}{\rightleftarrows}} \bullet f_2\ ,$$

where $I_A = 0$ *and* $I_B = \langle Z_{new}\rangle = \langle \eta^*\gamma^*\eta^*\rangle$.

- A direct computation shows that $HH^1(A) = 0$ and $HH^1(B) \simeq \langle \gamma^*\|\gamma^*\rangle$, hence $HH^1(B) \simeq HH^1(A) \times k$.
- Since $\delta_B^0(f_1\|\gamma^*\eta^*) = \gamma^*\|\gamma^*\eta^*\gamma^* = -\delta_B^0(f_2\|\gamma^*\eta^*)$, we have

$$Z(A) = \langle 1_A\rangle \hookrightarrow Z(B) \simeq \langle 1_B, f_1\|\eta^*\gamma^* + f_2\|\gamma^*\eta^*\rangle$$

Thanks! 谢谢大家!