Generalized parallel paths method with applications to Brauer graph algebras

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Parallel paths method

It is well-known that the Hochschild cohomology groups are important invariants of associative algebras. Since the Hochschild cohomology groups $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$, the computation of Hochschild cohomology groups are heavily based on a two-sided projective resolution of a given algebra A.

Parallel paths method

It is well-known that the Hochschild cohomology groups are important invariants of associative algebras. Since the Hochschild cohomology groups $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$, the computation of Hochschild cohomology groups are heavily based on a two-sided projective resolution of a given algebra A.

Based on the minimal two-sided projective resolution of monomial algebras, Strametz created the parallel paths method to compute the first Hochschild cohomology group in monomial cases.

Minimal projective resolution of monomial algebras **(Bardzell, 1997)**	$\Longrightarrow$	Parallel paths method of f.d. monomial algebras **(Strametz, 2006)**

Parallel paths method

Our aim is to generalize Strametz's parallel paths method on computing the first Hochschild cohomology groups from monomial algebras to general quiver algebras (may not be finite dimensional). We also give some applications on Brauer graph algebras.

Two-sided Anick resolution of quiver algebras **(Chen, Liu and Zhou, 2023)** $\Longrightarrow$ Parallel paths method of quiver algebras with finite reduced Gröbner basis **(Liu and Xing, 2023)**

Parallel paths method

We should point out that Rubio y Degrassi, Schroll and Solotar had done some analogous work independently by using the Chouhy-Solotar projective resolution which is constructed by reduction systems.

Chouhy-Solotar projective resolution of quiver algebras **(Chouhy and Solotar, 2015)**	$\Longrightarrow$	Parallel paths method of f.d. quiver algebras **(Rubio y Degrassi, Schroll and Solotar, 2022)**

Gröbner basis

Notations

k: a field,
$Q = (Q_0, Q_1)$: a finite quiver

- Two paths ε, γ of Q are called parallel if $s(\varepsilon) = s(\gamma)$ and $t(\varepsilon) = t(\gamma)$. An element in kQ is called uniform if it is a linear combination of parallel paths. If X and Y are sets of paths of Q, the set $X//Y$ is formed by the couples $(\varepsilon, \gamma) \in X \times Y$ such that ε and γ are parallel paths.
- We now fix an admissible well-order $>$ on $Q_{\geq 0}$. For any $a \in kQ$, we have $a = \sum_{p \in Q_{\geq 0},\ \lambda_p \in k} \lambda_p p$ and write $\mathrm{Supp}(a) = \{p \mid \lambda_p \neq 0\}$. We call $\mathrm{Tip}(a) = p$, if $p \in \mathrm{Supp}(a)$ and $p' \leq p$ for all $p' \in \mathrm{Supp}(a)$. Then we denote the tip of a set $W \subseteq kQ$ by $\mathrm{Tip}(W) = \{\mathrm{Tip}(w) | w \in W\}$

Gröbner basis

Gröbner basis of quiver algebras (Green, 1999)

We say $\mathcal{G}$ is a Gröbner basis for the ideal I with respect to the admissible order $>$ if $\mathcal{G}$ is a set of uniform elements in I such that

$$\langle \mathrm{Tip}(I) \rangle = \langle \mathrm{Tip}(\mathcal{G}) \rangle,$$

that is, $\mathrm{Tip}(I)$ and $\mathrm{Tip}(\mathcal{G})$ generate the same ideal in kQ.

Generalized parallel paths method

Notations

- Let ε be a path in Q and $(\alpha, \gamma) \in Q_1//Q$. Denote by $\varepsilon^{(\alpha,\gamma)}$ the sum of all nonzero paths obtained by replacing one appearance of the arrow α in ε by path γ. If the path ε does not contain the arrow α, we set $\varepsilon^{(\alpha,\gamma)} = 0$.

Generalized parallel paths method

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- Let $A = kQ/I$ be a quiver algebra with finite Gröbner basis $\mathcal{G}$. Then there is a k-linear basis $\mathcal{B}$ of the algebra A with respect to $\mathcal{G}$. Let the canonical projection be written as $\pi : kQ \to A$.

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- If X is a set of paths of Q and e a vertex of Q, the set Xe is formed by paths of X with source vertex e. In the same way eX denotes the set of all paths of X with terminus vertex e.

Generalized parallel paths method

Parallel paths method (Strametz, 2006)

Let $A = kQ/I$ be a finite dimensional monomial algebra with Z the minimal generating set of I. Then the beginning of the Hochschild cochain complex of A can be described by:

$$0 \longrightarrow k(Q_0//\mathcal{B}) \xrightarrow{\psi_0} k(Q_1//\mathcal{B}) \xrightarrow{\psi_1} k(Z//\mathcal{B}) \longrightarrow \cdots$$

where the differentials are given by

$$\begin{array}{rcl} \psi_0 & : & k(Q_0//\mathcal{B}) & \to & k(Q_1//\mathcal{B}), \\ & & (e,\gamma) & \mapsto & \sum_{\alpha\in Q_1 e}(\alpha,\pi(\alpha\gamma)) - \sum_{\beta\in eQ_1}(\beta,\pi(\gamma\beta)); \end{array}$$

$$\begin{array}{rcl} \psi_1 & : & k(Q_1//\mathcal{B}) & \to & k(Z//\mathcal{B}), \\ & & (\alpha,\gamma) & \mapsto & \sum_{p\in Z}(p,\pi(p^{(\alpha,\gamma)})). \end{array}$$

In particular, we have $\mathrm{HH}^0(A) \cong \mathrm{Ker}\psi_0$, $\mathrm{HH}^1(A) \cong \mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$.

Generalized parallel paths method

Let $A = kQ/I$ be a quiver algebra with the finite reduced Gröbner basis $\mathcal{G}$. The beginning of the Hochschild cochain complex of A can be described by:

$$0 \longrightarrow k(Q_0//\mathcal{B}) \xrightarrow{\psi_0} k(Q_1//\mathcal{B}) \xrightarrow{\psi_1} k(\mathrm{Tip}(\mathcal{G})//\mathcal{B}) \longrightarrow \cdots$$

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$$\begin{array}{rcll} \psi_1 & : & k(Q_1//\mathcal{B}) & \rightarrow & k(\mathrm{Tip}(\mathcal{G})//\mathcal{B}), \\ & & (\alpha,\gamma) & \mapsto & \sum_{g\in\mathcal{G}}\sum_{p\in\mathrm{Supp}(g)} c_g(p)\cdot(\mathrm{Tip}(g),\pi(p^{(\alpha,\gamma)})). \end{array}$$

with $g = \sum_{p\in\mathrm{Supp}(g)} c_g(p)p$, $c_g(p) \in k$. In particular, we have $\mathrm{HH}^0(A) \cong \mathrm{Ker}\psi_0$, $\mathrm{HH}^1(A) \cong \mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$.

Generalized parallel paths method

Lie structure

The bracket

$$[(\alpha,\gamma),(\beta,\varepsilon)] = (\beta,\pi(\varepsilon^{(\alpha,\gamma)})) - (\alpha,\pi(\gamma^{(\beta,\varepsilon)}))$$

for all $(\alpha,\gamma),(\beta,\varepsilon) \in Q_1//\mathcal{B}$ induces a Lie algebra structure on $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$, such that $\mathrm{HH}^1(A)$ and $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$ are isomorphic as Lie algebras.

Example

Example

Let char$k = 2$ and $A = k\langle x, y\rangle/\langle x^2, y^2, xy - yx\rangle$. Let $x > y$, then

$$\begin{array}{lllllllll}
\psi_0 & : & (1,1) & \mapsto & 0 & , & (1,x) & \mapsto & 0, \\
 & & (1,y) & \mapsto & 0 & , & (1,yx) & \mapsto & 0. \\
\psi_1 & : & (x,x) & \mapsto & 0 & , & (x,y) & \mapsto & 2(x^2,yx) = 0, \\
 & & (y,y) & \mapsto & 0 & , & (y,x) & \mapsto & 2(y^2,yx) = 0, \\
 & & (x,yx) & \mapsto & 0 & , & (x,1) & \mapsto & 2(x^2,x) + 2(xy,y) = 0, \\
 & & (y,yx) & \mapsto & 0 & , & (y,1) & \mapsto & 2(y^2,y) + 2(xy,x) = 0.
\end{array}$$

Then we get

$$\mathrm{HH}^1(A) = k\{(x,1),(y,1),(x,x),(y,x),(x,y),(y,y),(x,yx),(y,yx)\}.$$

...

Example

. . .

The Lie operations on $\mathrm{HH}^1(A)$ are

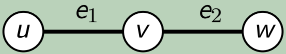
$$\begin{array}{lclclcl} [(x,1),(x,yx)] & = & (x,y) & , & [(y,1),(x,yx)] & = & (x,x) \\ [(x,1),(y,yx)] & = & (y,y) & , & [(y,1),(y,yx)] & = & (y,x) \\ [(x,1),(x,x)] & = & (x,1) & , & [(y,1),(y,y)] & = & (y,1) \\ [(y,y),(x,yx)] & = & (x,yx) & , & [(x,x),(y,yx)] & = & (y,yx) \end{array}$$

Therefore, $\mathrm{HH}^1(A) \cong sl_3(k)$ is a simple Lie algebra.

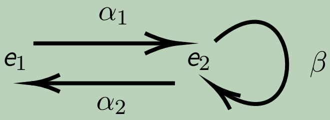
Applications to Brauer graph algebras

Example

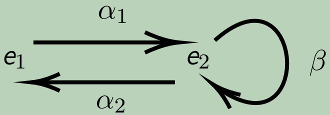
The Brauer graph $G = (V, E)$ is a finite (unoriented) connected graph:



with $m : V \to \mathbb{Z}_{>0}$ ($m(u) = m(v) = 1$, $m(w) = 3$), a multiplicity function of G. And there is an orientation o of G which is given by $o(u) : e_1$, $o(v) : e_1 < e_2 < e_1$, $o(w) : e_2 < e_2$. Then the quiver Q_G of G is given by:



Example



An ideal I_G in kQ_G generated by three types of relations.

- Relations of type I: $\beta^3 - \alpha_1\alpha_2$;
- Relations of type II: $\alpha_1\alpha_2\alpha_1$, $\alpha_2\alpha_1\alpha_2$, β^4;
- Relations of type III: $\beta\alpha_1$, $\alpha_2\beta$.

The corresponding Brauer graph algebra of Brauer G is

$$A = kQ_G/\langle \beta^3 - \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta \rangle.$$

Brauer graph

A Brauer graph G is a tuple $G = (V, E, m, o)$ where

- (V, E) is a finite (unoriented) connected graph.
- $m : V \to \mathbb{Z}_{>0}$ is a multiplicity function of G.
- o is called the orientation of G which is given, for every vertex $v \in V$, by a cyclic ordering of the edges incident with v.

Applications to Brauer graph algebras

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Quiver

Given a Brauer graph $G = (V, E, m, o)$, we can define a quiver $Q_G = (Q_0, Q_1)$ as follows: $Q_0 := E$,

$$Q_1 := \{i \to j \mid i, j \in E, \ \exists v \in V, \text{ such that } i < j \text{ belong to } o(v)\}.$$

For $i \in E$, if $v \in V$ is a vertex of i and i is not truncated at v, then there is a special i-cycle $C_v(\alpha)$ at v which is an oriented cycle given by $o(v)$ in Q_G with the starting arrow α (where the starting vertex of α in Q_G is i).

Applications to Brauer graph algebras

Brauer graph algebra (Donovan and Freislich, 1978)

We define an ideal I_G in kQ_G generated by three types of relations.

- Relations of type I: $C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')}$, for any $i \in Q_0$ and for any special i-cycles $C_v(\alpha)$ and $C_{v'}(\alpha')$ at v and v', respectively, such that both v and v' are not truncated.
- Relations of type II: $\alpha C_v(\alpha)^{m(v)}$, for any $i \in Q_0$, any $v \in V$ and where $C_v(\alpha)$ is a special i-cycle at v with starting arrow α.
- Relations of type III: $\beta\alpha$, for any $i \in Q_1$ such that $\beta\alpha$ is not a subpath of any special cycle except if $\beta = \alpha$ is a loop associated with a vertex v of valency one and multiplicity $m(v) > 1$.

The quotient algebra $A = kQ_G/I_G$ is called the Brauer graph algebra of the Brauer graph G.

Applications to Brauer graph algebras

The associated graded algebras

Let A be a finite dimensional algebra. Denote by $\mathfrak{r}$ the (Jacobson) radical $rad(A)$ of A. Then the graded algebra $gr(A)$ of A associated with the radical filtration is defined as follows. As a graded vector space,

$$gr(A) = A/\mathfrak{r} \oplus \mathfrak{r}/\mathfrak{r}^2 \oplus \cdots \oplus \mathfrak{r}^t/\mathfrak{r}^{t+1} \oplus \cdots .$$

The multiplication of $gr(A)$ is given as follows. For any two homogeneous elements:
$x + \mathfrak{r}^{m+1} \in \mathfrak{r}^m/\mathfrak{r}^{m+1}, \quad y + \mathfrak{r}^{n+1} \in \mathfrak{r}^n/\mathfrak{r}^{n+1}$, we have

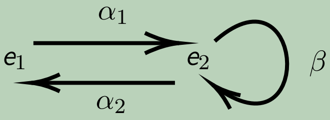
$$(x + \mathfrak{r}^{m+1}) \cdot (y + \mathfrak{r}^{n+1}) = xy + \mathfrak{r}^{m+n+1}.$$

Example

Consider the Brauer graph algebra

$$A = kQ_G/\langle \beta^3 - \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta \rangle$$

where Q_G is given by



Then the associated graded algebra of A is

$$gr(A) = kQ_G/\langle \beta^4, \alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta \rangle.$$

Applications to Brauer graph algebras

Denote by $val(v)$ the valency of the vertex $v \in V$. It is defined to be the number of edges in G incident to v. We call the edge i with vertex v truncated at v if $m(v)val(v) = 1$.

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Graded degree (Guo and Liu, 2021)

For each vertex v in a Brauer graph G, we define the graded degree $\mathrm{grd}(v)$ as follows. If $val(v) = 1$, we denote by v' the unique vertex adjacent to v. If G is given by a single edge with both vertices v and v' of multiplicity 1, then $grd(v) = grd(v') = 1$; Otherwise

$$grd(v) = \begin{cases} m(v)val(v), & \text{if } m(v)val(v) > 1; \\ grd(v'), & \text{if } m(v)val(v) = 1. \end{cases}$$

Applications to Brauer graph algebras

Balanced components

Let $G = (V, E)$ be a Brauer graph. We call an edge $v_1 \overset{i}{-} v_2$ in G with $grd(v_1) \neq grd(v_2)$ an unbalanced edge. Other edges which are not unbalanced will be called the balanced edges. We define the balanced components of G by the following rules:

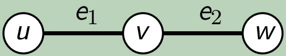
- retain the balanced edges in G;
- split the unbalanced edge into two edges by attaching two new truncated vertices.

The connected components in G after remodeling by the rules above are the balanced components of G. Denote the set of the balanced components of G by Γ_G.

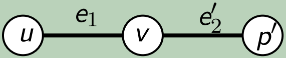
Applications to Brauer graph algebras

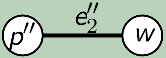
Example

Consider the Brauer graph $G = (V, E)$ which is given by



with $m(u) = m(v) = 1$, $m(w) = 3$. Then $grd(u) = grd(v) = 2$, $grd(w) = 3$. Thus e_1 is a balanced edge and e_2 is an unbalanced edge. The balanced components of G is given by





Actually, $grd(u) = grd(v) = grd(p') = 2$, $grd(p'') = grd(w) = 3$ and $|\Gamma_G| = 2$.

Applications to Brauer graph algebras

From now on, we assume that the characteristic of the ground field k is 0.

L_{00}

Let $A = kQ/I$, consider

$$L_0 := k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle,$$

which is a Lie subalgebra of $\mathrm{HH}^1(A)$. Furthermore, we can take L_{00} which is given by

$$L_{00} := \langle (\alpha, \alpha) | \alpha \in Q_1 \rangle \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle.$$

Actually, L_{00} is an abelian Lie subalgebra of $\mathrm{HH}^1(A)$ and $L_{00} \subseteq L_0$.

Applications to Brauer graph algebras

Example

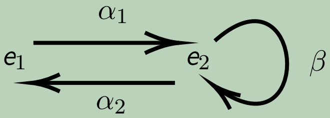
Consider

$$A = kQ_G/\langle \beta^3 - \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta\rangle,$$

and

$$gr(A) = kQ_G/\langle \beta^4, \alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta\rangle$$

where Q_G is given by



Applications to Brauer graph algebras

Example

By the parallel paths method,

$$\mathrm{HH}^1(A) = k\{3(\alpha_1, \alpha_1) + (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \alpha_1\alpha_2)\}$$

$$\mathrm{HH}^1(gr(A)) = k\{(\alpha_1, \alpha_1),\ (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \beta^3)\},$$

and

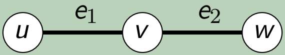
$$L_{00}^{A} = L_{0}^{A} = k\{3(\alpha_1, \alpha_1) + (\beta, \beta)\}$$

$$L_{00}^{gr(A)} = L_{0}^{gr(A)} = k\{(\alpha_1, \alpha_1),\ (\beta, \beta), \}.$$

Applications to Brauer graph algebras

Example

Recall the Brauer graph $G = (V, E)$ which is given by



Then

$$\dim_k L_{00}^{A} = 1 = 2 - 3 + 2 = |E| - |V| + 2,$$

and

$$\dim_k L_{00}^{gr(A)} = 2 = 2 - 3 + 1 + 2 = |E| - |V| + 1 + |\Gamma_G|.$$

Applications to Brauer graph algebras

Let A be a Brauer graph algebra with the corresponding Brauer graph G.

Lemma

$$\dim_k L_{00}^{A} = |E| - |V| + 2, \ \dim_k L_{00}^{gr(A)} = |E| - |V| + 1 + |\Gamma_G|.$$

Applications to Brauer graph algebras

Let A be a Brauer graph algebra with the corresponding Brauer graph G.

Lemma

$$\dim_k L_{00}^A = |E| - |V| + 2, \ \dim_k L_{00}^{gr(A)} = |E| - |V| + 1 + |\Gamma_G|.$$

Proposition (k: algebraically closed)

L_{00}^A (respectively, $L_{00}^{gr(A)}$) is a maximal diagonalizable subalgebra of $\mathrm{HH}^1(A)$ (respectively, $\mathrm{HH}^1(gr(A))$).

Denote the maximal torus of $\mathrm{Out}(A)^\circ$ (respectively, $\mathrm{Out}(gr(A))^\circ$) by $T(A)$ (respectively, $T(gr(A))$). Then the rank of $T(A)$ (respectively, $T(gr(A))$) is equal to the dimension of L_{00}^A (respectively, $L_{00}^{gr(A)}$).

Example

By the parallel paths method,

$$\mathrm{HH}^1(A) = k\{3(\alpha_1, \alpha_1) + (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \alpha_1\alpha_2)\}$$

$$\mathrm{HH}^1(gr(A)) = k\{(\alpha_1, \alpha_1),\ (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \beta^3)\}.$$

Then both $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(gr(A))$ are solvable. Moreover, there is a monomorphism $i : \mathrm{HH}^1(A) \to \mathrm{HH}^1(gr(A))$ as Lie algebras which is given by:

$$\begin{array}{rcl} i \ : \ 3(\alpha_1, \alpha_1) + (\beta, \beta) & \mapsto & 3(\alpha_1, \alpha_1) + (\beta, \beta), \\ (\beta, \beta^2) & \mapsto & (\beta, \beta^2), \\ (\beta, \alpha_1\alpha_2) & \mapsto & (\beta, \beta^3). \end{array}$$

Actually, $\dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) = 4 - 3 = \dim_k L_{00}^{gr(A)} - \dim_k L_{00}^{A} = 2 - 1 = |\Gamma_G| - 1.$

Let A be a Brauer graph algebra with the corresponding Brauer graph G.

Theorem

If G is different from $(\bullet = \bullet)$ (here both vertices have multiplicity 1), then both $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(gr(A))$ are solvable.

Applications to Brauer graph algebras

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Theorem

If $G \neq (v_S - v_L)$ with $m(v_L) > m(v_S) \geq 2$, then there is a monomorphism $i : \mathrm{HH}^1(A) \to \mathrm{HH}^1(gr(A))$ as Lie algebras.

Applications to Brauer graph algebras

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Corollary

$$\dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) = \dim_k L_{00}^{gr(A)} - \dim_k L_{00}^{A} = |\Gamma_G| - 1.$$

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THANKS !