Uniform Steiner bundles and adjoint pairs of reflection functors for Kronecker representations

Daniel Bissinger, Kiel University

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Motivation and overview, I

Let $\mathbb{k} = \overline{\mathbb{k}}$ field and $n \in \mathbb{N}$.

Interested in: Vector bundles (locally free sheaves) on $\mathbb{P}^n = \mathbb{P}^n(\mathbb{k})$.

Important technique: Study behavior of Vbs along pullbacks.

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$$\mathcal{F}|_L \cong \bigoplus_{i \in \mathbb{Z}} a_i(L) \underbrace{\mathcal{O}_L(i)}_{\textit{Serre twist}}. \quad \text{(Grothendieck, 1975)}$$

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$\rightsquigarrow \mathcal{F}|_{\mathbb{P}^1} = \bigoplus_{i\in\mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ (splitting type) & $\mathrm{supp}(\mathcal{F}) := \{i | a_i \neq 0\}$.

Motivation and overview, II

Definition (Dolgachev-Kapranov, 1993)

A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ is called Steiner bundle if there exist vector spaces V_1, V_2 and a short exact sequence
$0 \to V_1 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^n}(-1) \to V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0.$

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Theorem (Marchesi-Miró-Roig, 2022)

Let $k \in \mathbb{N}$, $\Bbbk = \mathbb{C}$. There exist uniform (and non-homogeneous) Steiner bundles $\mathcal{F}$ of k-type, i.e. $\mathcal{F}|_{\mathbb{P}^1} \cong \bigoplus_{i=0}^{k} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ with $a_k \neq 0$.

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Remark

1. *All known uniform Steiner bundles are of connected type, i.e. the support is an intervall in $\mathbb{Z}$.*
2. *All given examples contain 0 in the support.*

This talk: Construct new uniform Steiner bundles using rep. theory of the Kronecker quiver

$$K_r := \quad 1 \underset{\gamma_r}{\overset{\gamma_1}{\rightrightarrows}} 2 \quad (\text{arrows } \gamma_1, \dots, \gamma_r)$$

with arrow space $A_r := \bigoplus_{i=1}^{r} \Bbbk\gamma_i$.

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- $M \in \text{rep}(K_r) \triangleq \psi_M \colon A_r \otimes_{\mathbb{k}} M_1 \to M_2; \gamma_i \otimes m \mapsto M(\gamma_i)(m)$.
- $\text{rep}_{\text{inj}}(K_r, 1) := \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\mathbb{k}} M_1} = \{0\} \ \forall \, \mathfrak{v} \in \mathbb{P}(A_r)\}$ is the space 1-injective representations.

Motivation and overview, IV

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There is a right exact functor $\widetilde{\Theta}_r \colon \mathrm{rep}(K_r) \to \mathrm{Coh}(\mathbb{P}(A_r))$ *s.t. the following statements hold.*

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Let $\alpha \in \text{Inj}_{\mathbb{k}}(A_1, A_r)$, then

$$\alpha^*(M) \cong a_\alpha \underbrace{(\mathbb{k} \overset{\text{id}}{\to} \mathbb{k})}_{=P(1)} \oplus b_\alpha \underbrace{(0 \to \mathbb{k})}_{=P(2)} \oplus c_\alpha (\mathbb{k} \to 0) \in \text{rep}(K_1), \text{ and}$$

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Restrictions to K_2, I

Question: Can we study restrictions to K_2 via $\mathrm{Hom}_{K_r}(-,-)$?

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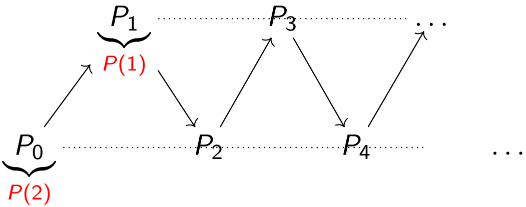
- Preprojective Auslander-Reiten component



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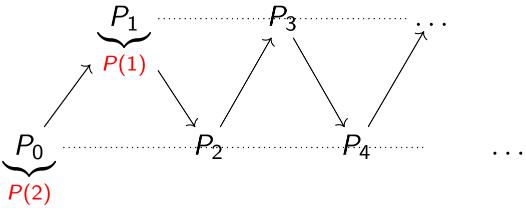

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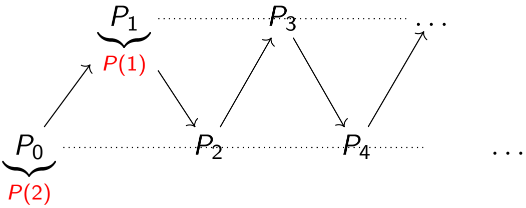

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$M \in \mathrm{rep_{inj}}(K_r, 1)$, $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then $\alpha^*(M) \in \mathrm{rep_{inj}}(K_2, 1)$ and

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There exist representations $(X(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_2(A_r)}$ *s.t. for all* $M \in \mathrm{rep}(K_r)$

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$M \in \operatorname{rep}_{\mathrm{inj}}(K_r, 1)$, $\alpha \in \operatorname{Inj}_{\Bbbk}(A_2, A_r)$, then $\alpha^*(M) \in \operatorname{rep}_{\mathrm{inj}}(K_2, 1)$ and

$$\alpha^*(M) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) P_n.$$

For $L := \mathbb{P}(\operatorname{im}\alpha) \subseteq \mathbb{P}(A_r)$ we obtain

$$\widetilde{\Theta}_r(M)|_L \cong \widetilde{\Theta}_2(\alpha^*(M)) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha)\widetilde{\Theta}_2(P_n) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha)\mathcal{O}_{\mathbb{P}(A_2)}(n).$$

Proposition (B.-Farnsteiner, 2022)

There exist representations $(X(\mathfrak{v}))_{\mathfrak{v} \in \operatorname{Gr}_2(A_r)}$ *s.t. for all* $M \in \operatorname{rep}(K_r)$

1. $\operatorname{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = 2 \cdot \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(X(\mathfrak{v}), M)$, *and*
2. $\alpha^*(M) \in \operatorname{rep}(K_2)$ *is projective iff* $\ker \psi_M|_{\operatorname{im}\alpha \otimes_{\Bbbk} M_1} = \{0\}$.

Restrictions to K_2, III

Corollary

Let $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) := \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} = \{0\} \ \forall\, \mathfrak{v} \in \mathrm{Gr}_2(A_r)\}$ *be 2-injective with* $\dim_{\Bbbk} M_1 \neq 0$.

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We have $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$.

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We have $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$. Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then

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Steiner bundles arrange nicely in AR components

For $r \geq 3$ and $\mathcal{C}$ regular AR component there are $M_{\mathcal{C},1}, M_{\mathcal{C},2} \in \mathcal{C}$ quasi-simple s.t.

$$\mathcal{C} \cap \mathrm{rep}_{\mathrm{inj}}(K_r, i) = (M_{\mathcal{C},i} \to) \quad (\text{cone of successors}).$$

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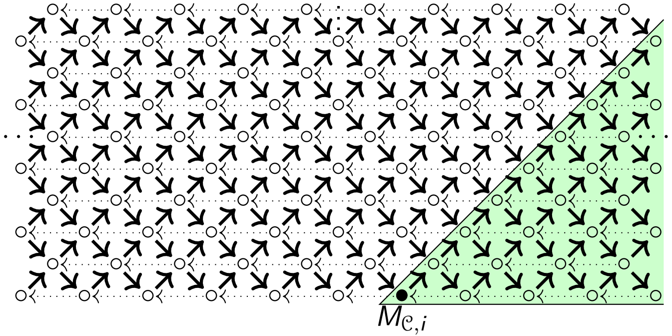
Illustration:



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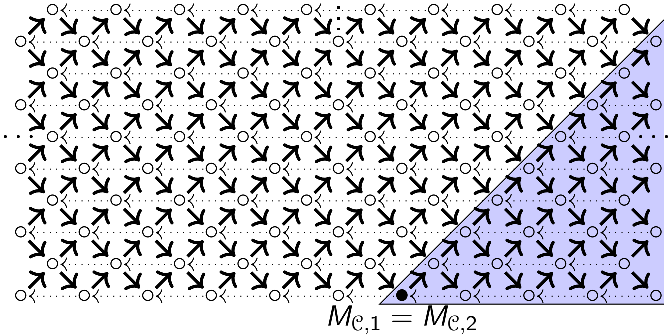

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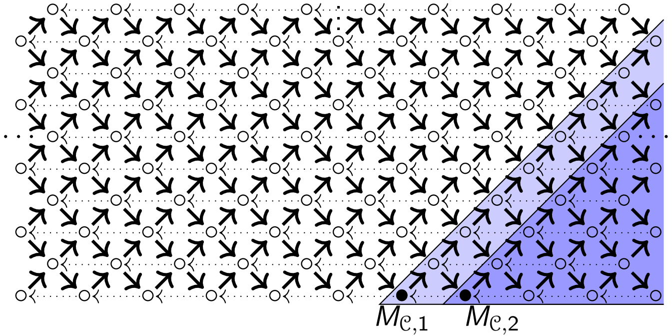

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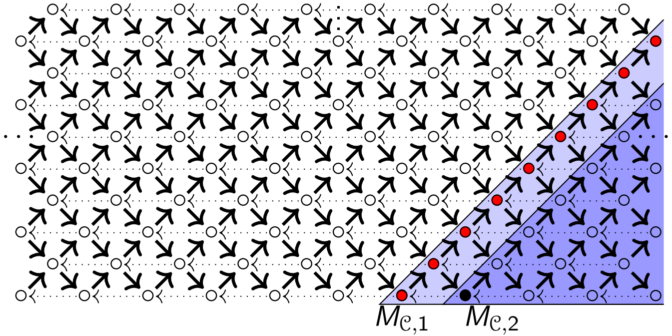
Case 2:

Application of $\widetilde{\Theta}_r$: interesting bundles



Restriction, inflation and reflection, I

- Consider $\iota\colon A_2 \hookrightarrow A_r; \gamma_i \mapsto \gamma_i$ and let

$$\mathrm{res}\colon\ \mathrm{rep}(K_r) \to \mathrm{rep}(K_2); \psi_M \mapsto \psi_{\iota^*(M)} \text{ (restriction)}.$$

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- Let
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$$\psi_X \mapsto \psi_{\mathrm{inf}(X)}(\gamma_i \otimes x) = \begin{cases} \psi_X(\gamma_i \otimes x), & i \in \{1,2\} \\ 0, & \textit{else}. \end{cases}$$

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- Let
$$\sigma_{K_d}\colon\ \mathrm{rep}(K_d) \to \mathrm{rep}(K_d)\ (\text{reflection})$$
$$\psi_{\sigma_{K_d}}\colon A_d \otimes_{\Bbbk} \ker\psi_M \to M_1; \gamma_i \otimes (\gamma_j \otimes m) \mapsto \delta_{ij} \cdot m.$$

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Proposition (B. 2024)

The functor $\sigma_{K_r}^{-1} \circ \inf\colon \mathrm{rep}(K_2) \to \mathrm{rep}(K_r)$ *is left adjoint to* $\sigma_{K_2} \circ \mathrm{res}\colon \mathrm{rep}(K_r) \to \mathrm{rep}(K_2)$.

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1. $\sigma_{K_2}^{-1}(P_n) = P_{n+1}$ for all $n \in \mathbb{N}_0$.

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- Similarly, $\dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(X_n, M) = \sum_{i=0}^{n-1} a_i (n-i)$.

Restriction, inflation and reflection, III

Theorem (B. 2024)

Let $n \in \mathbb{N}$. There exist representations $(X_n(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_2(A_r)}$ s.t. for all $M \in \mathrm{rep_{inj}}(K_r, 1)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$

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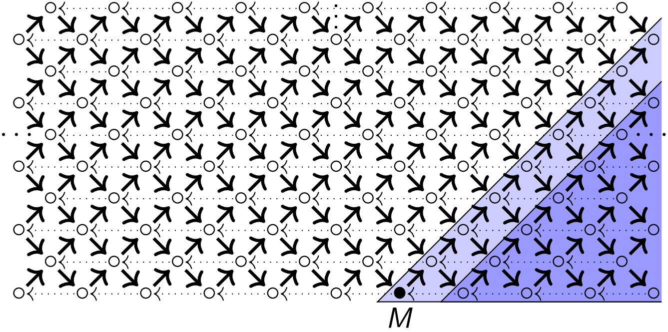

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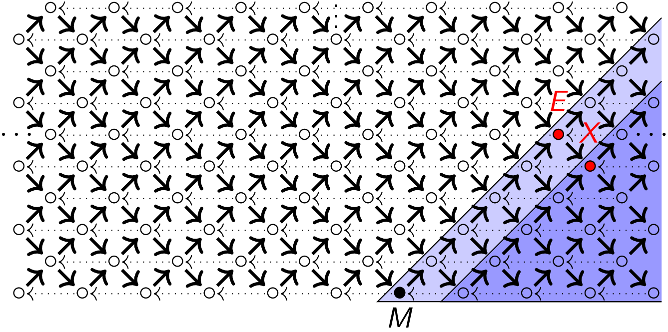

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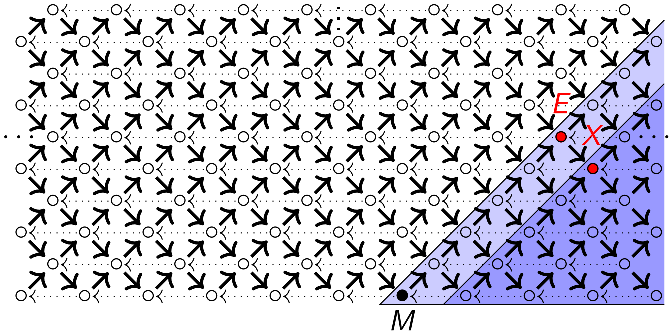

$$\Longrightarrow \quad 0 \to M \to E \to X \to 0.$$

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Corollary

Let $n \in \mathbb{N}$. There exist $c_0, c_1, c_n, c_{n+1} \in \mathbb{N}$ and an indecomposable and uniform Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ such that

$$\mathcal{F}|_{\mathbb{P}(A_2)} \cong c_0 \mathcal{O}_{\mathbb{P}(A_2)} \oplus c_1 \mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus c_n \mathcal{O}_{\mathbb{P}(A_2)}(n) \oplus c_{n+1} \mathcal{O}_{\mathbb{P}(A_2)}(n+1).$$

Application, II

Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then

$$0 \to \alpha^*(M) \to \alpha^*(E) \to \alpha^*(X) \to 0.$$

Since $X \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$, $\alpha^*(X)$ is projective and

$$\alpha^*(E) \cong a_n P_n \oplus b_n P_{n+1} \oplus \Delta_X(2) P_0 \oplus \dim_{\Bbbk} X_1 P_1$$

Corollary

Let $n \in \mathbb{N}$. There exist $c_0, c_1, c_n, c_{n+1} \in \mathbb{N}$ and an indecomposable and uniform Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ such that

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$\rightsquigarrow$ New indecomposable uniform Steiner bundles with disconnected splitting types (having arbitrarily wide gaps).

Thank you for your attention!