Cluster algebras, quantum affine algebras, and categorifications

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August 8, 2024

Motivation I

Cluster algebras

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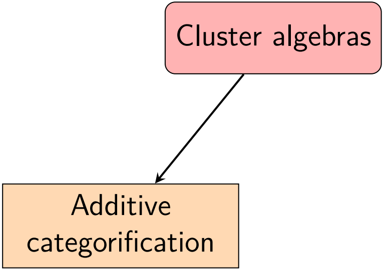

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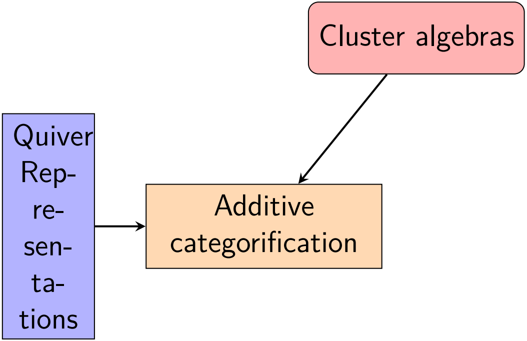

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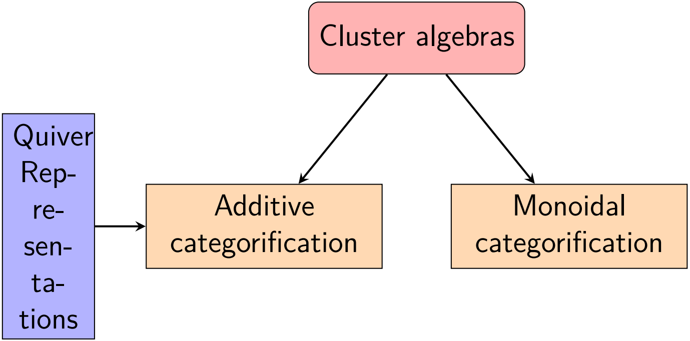

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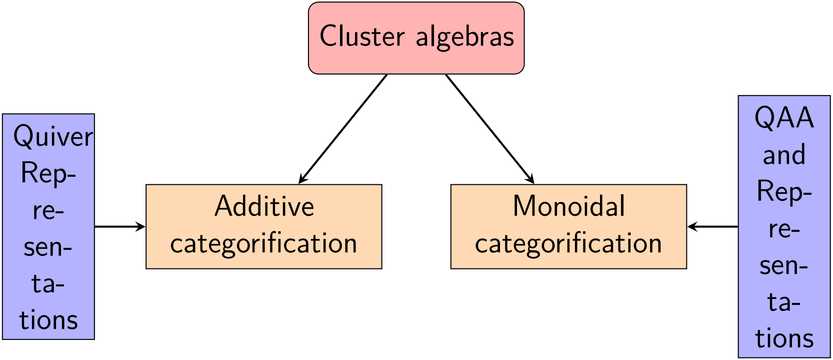

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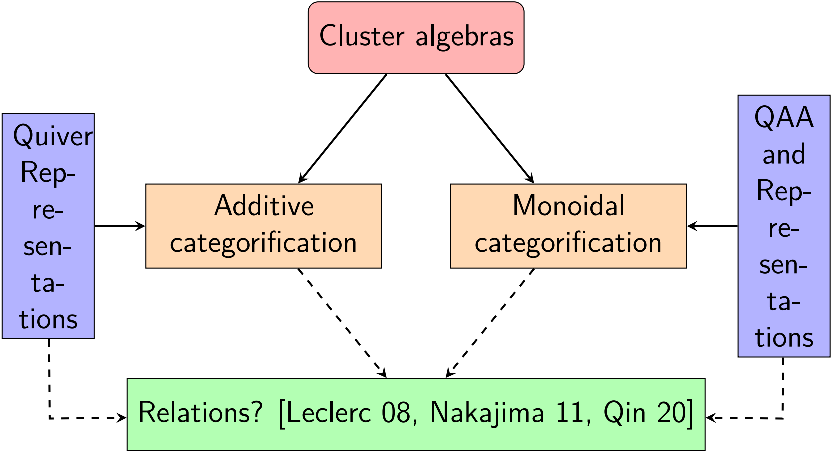

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- $U_q(\widehat{\mathfrak{g}})$ = the quantum affine algebra associated to $\mathfrak{g}$, where q is not a root of unity, with generators $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,s}$ ($i \in I$, $s \in \mathbb{Z} \backslash \{0\}$), and central element $c^{\pm 1}$, subject to certain relations.

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- $\mathcal{P}$ = the free ablian group generated by $Y_{i,a}^{\pm 1}$, with $i \in I$, $a \in \mathbb{C}^*$.
- $\mathcal{P}^+$ = the submonoid of $\mathcal{P}$ generated by $Y_{i,a}$, with $i \in I$, $a \in \mathbb{C}^*$. Elements in $\mathcal{P}^+$ are called dominant monomials.

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- $L(m)$ = the simple module with highest l-weight monomial $m \in \mathcal{P}^+$.
- $\mathscr{C}$ = the abelian monoidal category of f.d. $U_q(\widehat{\mathfrak{g}})$-modules (type 1).
- $K_0(\mathscr{C})$ = the Grothendieck ring of $\mathscr{C}$.
- A simple $U_q(\widehat{\mathfrak{g}})$-module M is real if $M \otimes M$ is simple, otherwise M is imaginary. A simple $U_q(\widehat{\mathfrak{g}})$-module M is prime if it admits no non-trivial tensor decomposition.

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- In $U_q(\widehat{\mathfrak{sl}_2})$ case, simple = real. Unfortunately, the $U_q(\widehat{\mathfrak{sl}_2})$-philosophy is not true for general rank. For example, $L(Y_{1,-3}Y_{2,-6}Y_{2,0}Y_{3,-3})$ is an imaginary module.

Hernandez and Leclerc's work

Hernandez–Leclerc[1] proved that $K_0(\mathscr{C}_{\mathbb{Z}}^-)$ is a cluster algebra, and conjectured

Hernandez–Leclerc Conjecture

A simple module is real if and only if it is a cluster monomial.

Geometric q-character formula Conjecture

The standard truncated q-chracter of a real module is the F-polynomial of a generic kernel over some Jacobi algebra.

[1] D. Hernandez, B. Leclerc, A cluster algebra approach to q-characters of Kirillov-Reshetikhin modules, J. Eur. Math. Soc. 18 (5) (2016)

Open part of Hernandez–Leclerc Conjecture

Following Hernandez–Lelcerc's work, we know that

Hernandez–Leclerc Conjecture $\Rightarrow$ Geometric q-character formula Conjecture

Qin, independently Kashiwara–Kim–Oh–Park, proved that cluster monomials are real modules.

The difficult (still open) part of Hernandez–Leclerc Conjecture is that

Reachability of real modules (Qin)

Real modules are reachable/cluster monomials.

In order to solve their conjectures, Hernandez and Leclerc[2] proposed the following open question:

The classification of real modules (Hernandez–Leclerc)

How to classify real modules in terms of their highest l-weight monomials?

[2] D. Hernandez, B. Leclerc, *Quantum affine algebras and cluster algebras*, Interactions of quantum affine algebras with cluster algebras, current algebras and categorification–in honor of Vyjayanthi Chari on the occasion of her 60th birthday, Progr. Math., 337, Birkhäuser/Springer, Cham, 2021, 37–65.

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by establishing an interaction

between additive categorification

and monoidal categorification.

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$$(i, r) \to (j, s) \Leftrightarrow c_{ij} \neq 0, s = r + c_{ij}.$$

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- Choose one of two isomorphic connected components, denoted by Γ, with vertex set V.
- $\Gamma^{\leq 0}$ = the semi-infinite subquiver of Γ with vertex set $V \cap (I \times \mathbb{Z}_{\leq 0})$, called an Hernandez–Leclerc quiver.

Examples

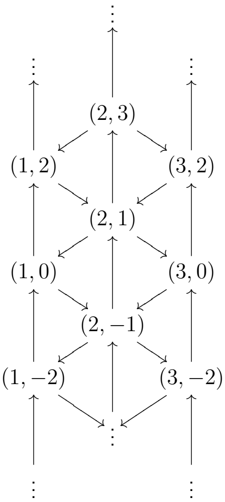

Figure: Quivers Γ (left) and $\Gamma^{\leq 0}$ (right) in type A_3.

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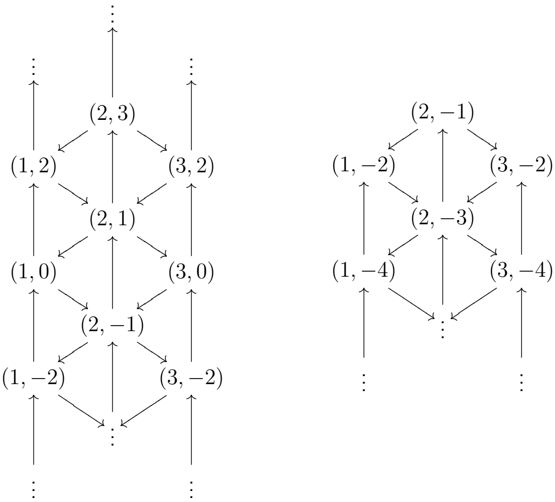

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Quiver $\Gamma^{\leq\xi}$

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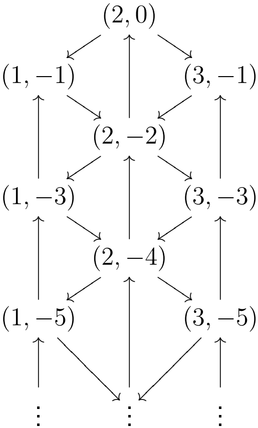

Figure: In type A_3, $\xi(1,2,3) = (-1,0,-1)$ (left) and $\xi(1,2,3) = (0,-1,0)$ (right).

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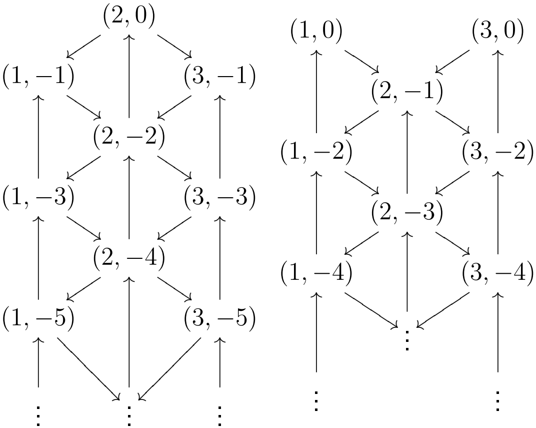

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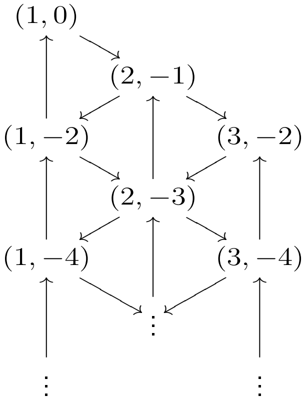

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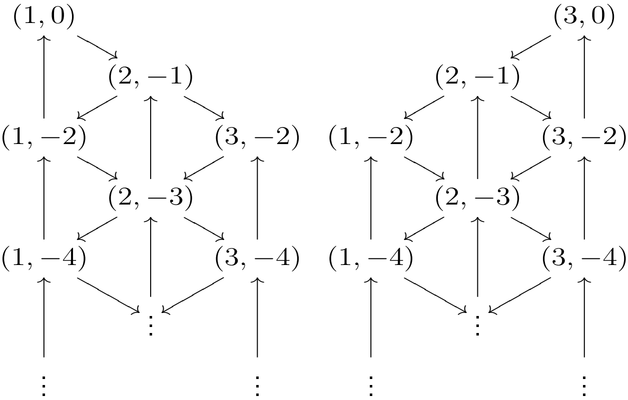
These four quivers are all possibilities of $\Gamma^{\leq \xi}$ up to shift of parameter.



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where the highest ℓ-monomials of composition factors of any object in $\mathscr{C}_\ell^{\leq \xi}$ are monomials in $Y_{i,\xi(i)-2k}$ for $i \in I, 0 \leq k \leq \ell$.

Theorem (DS23)

The quantum Grothendieck ring $K_t(\mathscr{C}_\ell^{\leq \xi})$ of $\mathscr{C}_\ell^{\leq \xi}$ admits a quantum cluster algebra structure. The isomorphism is given by the truncated (q,t)-characters.

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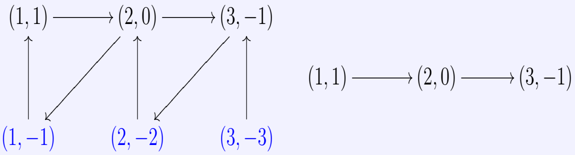

Figure: Quiver of $K_t(\mathscr{C}_1^{\leq \xi})$ in A_3, $\xi(1,2,3) = (1,0,-1)$.

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- Let $\mathcal{C}_1^{\leq \xi}$ be the cluster category of $A_1^{\leq \xi}$, which is a triangulated 2-Calabi-Yau category with cluster-tilting objects.

For each pair of non-split triangles

$$L \to M \to N \to L[1], \quad N \to M' \to L \to N[1] \qquad (1)$$

in $\mathcal{C}_1^{\leq \xi}$, with L and N indecomposable, $g(M) = g(L) + g(N)$ and $\mathrm{Ext}^1_{\mathcal{C}_1^{\leq \xi}}(L, N) = 1$, we have

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where

- X_L is the cluster variable associated to L,
- $\mathbf{y}^{\alpha}$ is a Laurent monomial in variables y_i, with $i \in I$, and
- the vector $\alpha \in \mathbb{Z}^I$ is the dimension vector of the image of the morphism $h : \tau_{\mathcal{C}}^{-1} L \to N$ from the second exchangeable triangle.

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For $i \in I$, let $f_i = Y_{i,\xi(i)-2} Y_{i,\xi(i)}$, and the modules

$$L(f_i) = L(Y_{i,\xi(i)-2} Y_{i,\xi(i)})$$

are Kirillov-Reshetikhin modules.

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Let (L, N) be an exchange pair in $\mathcal{C}_1^{\leq \xi}$ with exchange triangles (1). We define

$$\kappa(L, M, N) = \underline{\dim}(\mathrm{soc}(\mathscr{F} L)) + \underline{\dim}(\mathrm{soc}(\mathscr{F} N)) - \underline{\dim}(\mathrm{soc}(\mathscr{F} M)).$$

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Similarly, we define $\kappa(L, M', N)$.

Theorem (DS23)

Let (L, N) *be an exchange pair in* $\mathcal{C}_1^{\leq \xi}$ *with* $N \in \operatorname{mod} kA_1^{\leq \xi}$ *and with exchange triangles (1). Then there is an exact sequence in* $\mathscr{C}_1^{\leq \xi}$

$$0 \to \Phi(M') \otimes \left(\bigotimes_{j \in I} L(f_j)^{\otimes d_j} \right) \to \Phi(L) \otimes \Phi(N) \to \Phi(M) \otimes \left(\bigotimes_{i \in I} L(f_i)^{\otimes c_i} \right) \to 0, \tag{2}$$

Theorem (DS23)

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or the same sequence with arrows reversed, where

$$(c_i)_{i \in I} = \kappa(L, M, N), (d_j)_{j \in I} = \kappa(L, M', N) + g(\operatorname{Im}(h)).$$

In particular, $\mathbf{hw}(\Phi(L))\mathbf{hw}(\Phi(N)) = \mathbf{hw}(\Phi(M)) \left(\prod_{i \in I} f_i^{c_i}\right)$.

Mesh relations I

- If N is an indecomposable non-projective module and $L = \tau N$, then $M' = 0$ and $\alpha = \underline{\dim}(N)$. In this case, we obtain the following exchange relation:

$$[\Phi(\tau N)][\Phi(N)] = [\Phi(M)] \left(\prod_{i \in I} [L(f_i)]^{c_i} \right) + \prod_{j \in I} [L(f_j)]^{d_j}.$$

Mesh relation II

- In particular, for $L = P(i)[1]$, $N = I(i)$, we consider the following non-split exchange triangles

$$P(i)[1] \to 0 \to I(i) \to P(i)[2] = I(i),$$

$$I(i) \to \left(\bigoplus_{j: i \to j} P(j)[1] \right) \oplus \left(\bigoplus_{j: j \to i} I(j) \right) \to P(i)[1],$$

our theorem yields certain equations from T-systems.

$$[\Phi(P(i)[1])][\Phi(I(i))] = [L(f_i)] + \prod_{j: i \to j} [\Phi(P(j)[1])] \prod_{j: j \to i} [\Phi(I(j))].$$

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- HL-modules of type A were studied and named by Brito and Chari. Our method (for type A) is different with Brito–Chari's. Moreover, HL-modules of type D, E_6, E_7, E_8 are new.

For $\ell \neq 1$

- Our method is valid for reachable real prime modules. In particular, we explicitly classify real prime modules for any height function ξ, in the cases where
  - $\ell \leq 4$ in type A_2, and
  - $\ell = 2$ in type A_3, A_4.

  In the cases, we use the representation theory of cluster-tilting algebras.

Thank you for your listening!