Deformation theory via reduction systems & applications

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Deformation theory — why?

- understand variation of a known object
  algebra, module, category
- produce new objects from known ones
  a priori different, but often similar
- related to quantization in quantum physics
  (needs algebraic as well as analytic tools)

Deformation theory — how?

Abstract answer

Controlled by higher structures on cochain complexes

DG Lie / L_∞ algebra

Pridham, Lurie

Ex. $\mathrm{Hom}(\mathrm{Bar}_\bullet(A), A)$ + Gerstenhaber bracket

Hochschild complex

controls deformations of

- associative algebras
- Abelian / DG enhanced triangulated categories

Keller, Lowen-Van den Bergh, Blanc-Katzarkov-Pandit

Fundamental questions

- how to describe deformations explicitly?
- how to pass from formal to "strict" deformations?

Planck's constant

$\hbar \longrightarrow 1$

$\frac{1}{c} \longrightarrow 1$

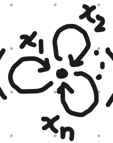
speed of light

Deformation theory — how?

Concrete answer for $A = \Bbbk Q/I$

Main idea Replace $\mathrm{Bar}_\bullet(A)$ by smaller resolution

Reduction systems for $\Bbbk Q/I$ $\supset$ Gröbner–Shirshov bases
a/k/a noncomm. Gröbner bases

Ex. $A = \Bbbk[x_1,\dots,x_n] = \Bbbk\langle$  $\rangle / \text{comm.}$

$$R = \{(\underbrace{x_j x_i}_{\text{leading term}}, x_i x_j)\}_{1\le i<j\le n}$$

__reduction__ replace $\underbrace{x_j x_i}$ by $x_i x_j$ for $i<j$

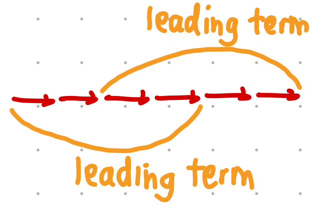
Diamond Condition

$x_{i_1} x_{i_2} \cdots x_{i_m}$ reduces to $x_1^{k_1} \cdots x_n^{k_n}$ irrespective of order of reductions

reductions always terminate after fin. many steps

Diamond Lemma If R is __reduction-finite__ then TFAE

1. R satisfies the Diamond Condition on all paths
2. —————— " —————— overlaps
3. irreducible paths form a $\Bbbk$-basis of $A = \Bbbk Q / \underbrace{\langle s-t\rangle_{(s,t)\in R}}_{I}$



$x_k x_j x_i$

$i<j<k$

Reduction systems $\longrightarrow$ resolutions

$$\boxed{\begin{array}{c} A = \Bbbk Q / I \\ \text{algebra} \end{array}} \longrightarrow \boxed{\begin{array}{c} R \\ \text{reduction system} \end{array}} \longrightarrow \boxed{\begin{array}{c} P_\bullet \\ \text{projective bimodule resolution of } A \end{array}}$$

word combinatorics
of higher overlaps

Bardzell, Chouhy–Solotar

Thm. [B–Wang] There are explicit homotopy deformation retracts

$$\begin{tikzcd} P_\bullet \arrow[r, shift left, "F_\bullet"] & \mathrm{Bar}_\bullet(A) \arrow[l, shift left, "G_\bullet"] \arrow[loop right, "h"] \end{tikzcd} \qquad \xrightarrow{\ \mathrm{Hom}(-,A)\ } \qquad \begin{tikzcd} \mathrm{Hom}(P_\bullet, A) \arrow[r, shift left] & \underbrace{\mathrm{Hom}(\mathrm{Bar}_\bullet(A), A)}_{\text{Hochschild complex}} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

The Homotopy Transfer Theorem yields (after several steps)

Thm. [B–Wang] There is an equivalence between

- formal deformations of A
- formal deformations of R $\rightsquigarrow$ explicit deformed product

controlled by $p(Q,R) = (\mathrm{Hom}(P_\bullet, A), \partial, \langle -,- \rangle, \ldots)$

L_∞ algebra

Rmk. (1.) Every formal deformation of A is gauge equivalent to $(A[\![t]\!], \star)$

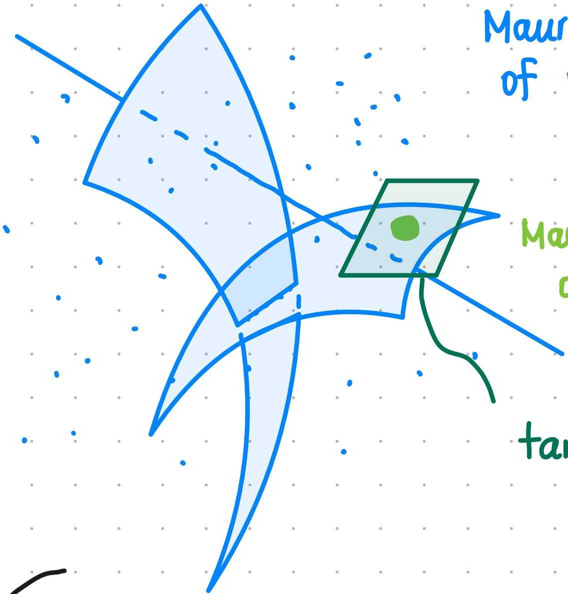
explicit formula

(2.) $HH^2(A,A) \simeq$ first-order def. of $R/\text{equiv.}$

Geometric picture

$\mathrm{Hom}\left(\Bbbk\left\{\begin{smallmatrix}\text{leading}\\ \text{terms}\end{smallmatrix}\right\}, A\right)$

$\cup$



Maurer–Cartan elements of $P(Q,R)$ — strict

Maurer-Cartan elements of $P(Q,R)\hat{\otimes}\mathfrak{m}$ — formal

tangent space $Z^2(P^\bullet)$
$$\downarrow$$
$HH^2(A,A)$ — first-order

groupoid action $\longleftrightarrow$ equivalence of reduction systems

Applications ⇝ new avenues

[B-Wang] Explicit description of Abelian deformations of $\mathrm{Qcoh}(X)$ for X any Noetherian scheme of finite type $/\Bbbk$

Q Describe Gabriel spectrum of deformation of $\mathrm{Qcoh}(X)$

[B-Schmitt] Strict deformation quantization of polynomial Poisson structures on $\mathbb{R}^d$. completion w.r.t. to locally convex topology on $\mathbb{C}[x_1,\dots,x_d]$

Q Explicit rational star products?

[B-Wang] A_∞ deformations of extended Khovanov arc algebras & solution of Stroppel's Conjecture (ICM 2010)

Q New knot invariants from A_∞ deformations?

[B-Schroll-Wang] A_∞ deformations of partially wrapped Fukaya categories of surfaces

Zhengfang Wang's talk

Q New classes of associative algebras closed under derived equivalence?

use Chouhy-Solotar resolution for cofibrant replacement

Thank you!