w-operation on the Anderson rings

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Notation

- R is a commutative ring with identity.
- D is an integral domain with quotient field K.
- K is a field.
- $R[X], D[X], K[X]$ are the polynomial rings over R, D, K, respectively.
- $\mathrm{Max}(R)$ is the set of maximal ideals of R.
- $\mathrm{Spec}(R)$ is the set of prime ideals of R.
- $\mathbf{F}(D)$ is the set of nonzero fractional ideals of D.
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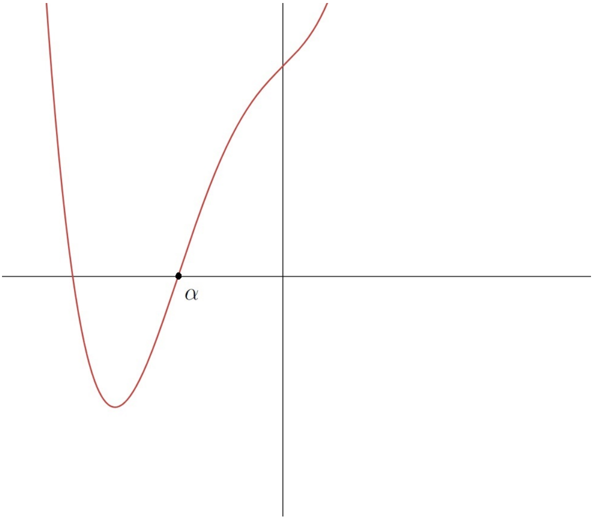
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- $$\begin{aligned} K[X] \setminus M_0 &= \{f \in K[X] \mid f(0) \neq 0\} \\ &= \{f \in K[X] \mid f(0) \text{ is a unit in } K\}. \end{aligned}$$

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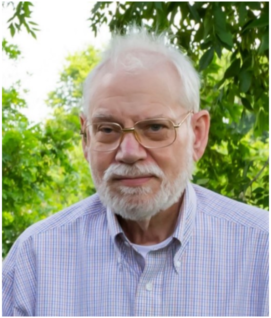
In this case, $R[X]_A$ is called the *Anderson ring* of R.



Daniel D. Anderson (1948 - 2022)

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Nagata rings and Serre's conjecture rings

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Consider the multiplicative sets of $R[X]$

$$N = \{f \in R[X] \mid c(f) = R\} \text{ and } U = \{f \in R[X] \mid f \text{ is monic}\}$$
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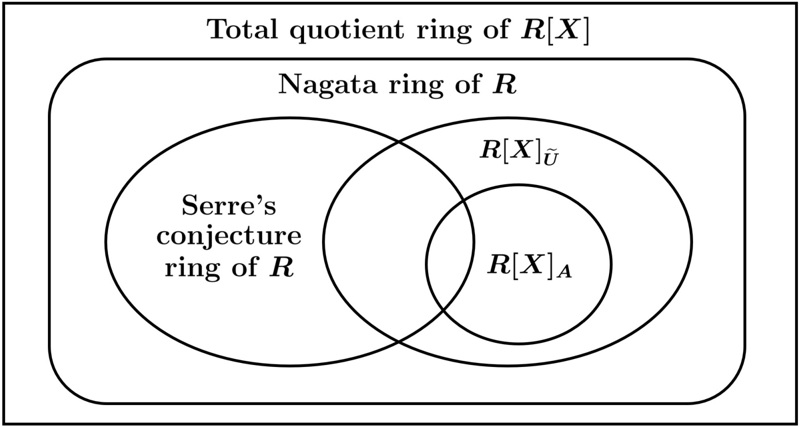

Maximal ideals of the Anderson rings

Theorem (Preprint, Baek and Lim)

Let R be a commutative ring with identity.
Then there is a one-to-one correspondence between
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In fact, $\mathrm{Max}(R[X]_A) = \{(M + XR[X])_A \mid M \in \mathrm{Max}(R)\}$.

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- $R[X]_A$ has finite character if and only if R is a semi-quasi-local ring.

	$R[X]_N$	$R[X]_U$	$R[X]_A$
Maximal ideal extension	O	O	X
Maximal ideal correspondence	O	X	O

Localization of the Anderson rings at their maximal ideals

Lemma (Preprint, Baek and Lim)

Let R be a commutative ring with identity.
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- If R is a quasi-local ring, then $A = R[X] \setminus (M + XR[X])$.
- If $\mathfrak{m} \in \mathrm{Max}(R[X]_A)$, then $(R[X]_A)_{\mathfrak{m}} = R_M[X]_{A_M}$, where $A_M = \{f \in R_M[X] \mid f(0) \text{ is a unit in } R_M\}$ for some $M \in \mathrm{Max}(R)$.

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- R_M has a property (P) for all $M \in \mathrm{Max}(R)$ if and only if $(R[X]_A)_{\mathfrak{m}}$ has a property (P) for all $\mathfrak{m} \in \mathrm{Max}(R[X]_A)$.

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Example

R is a locally Noetherian ring if and only if
$R[X]_A$ is a locally Noetherian ring.

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Then a map $* : \mathbf{F}(D) \to \mathbf{F}(D)$ given by $F \mapsto F_*$ is a *star-operation* if for $F, F_1, F_2 \in \mathbf{F}(D)$, $k \in K \setminus \{0\}$,

(1) $(k)_* = (k)$ and $(kF)_* = kF_*$

(2) if $F_1 \subseteq F_2$, then $(F_1)_* \subseteq (F_2)_*$

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- An ideal I of D is a *$*$-ideal* of D if $I_* = I$.
- An ideal I of D is a *maximal $*$-ideal* of D if there is no $*$-ideal properly containing I.
- $*\text{-}\mathrm{Max}(D)$ is the set of maximal $*$-ideal of D, and called the *$*$-maximal spectrum* of D.

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- For each $F \in \mathbf{F}(D)$, *w-envelop* of F is the set $F_{w_D} := \{x \in F \otimes K \mid xJ \subseteq F \text{ for some } J \in GV(D)\}$.

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- $w\text{-}\mathrm{Max}(D) \neq \emptyset$

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Basic properties of star-operations on the Anderson rings

Proposition (Preprint, Baek and Lim)

Let D be an integral domain and let I be a nonzero fractional ideal of D. Then the following assertions hold.

(1) $(ID[X]_A)^{-1} = I^{-1}D[X]_A$.

(2) $(ID[X]_A)_v = I_vD[X]_A$.

(3) $(ID[X]_A)_t = I_tD[X]_A$.

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- For $* = v, t$ or w, then I is a $*$-ideal if and only if $ID[X]_A$ is a $*$-ideal.

w-Maximal spectrum of the Anderson rings

Theorem (Preprint, Baek and Lim)

Let D be an integral domain.
If $\mathfrak{m}$ is a maximal w-ideal of $D[X]_A$, then $\mathfrak{m}$ is exactly of the form

(1) $MD[X]_A$ for some maximal w-ideal M of D, or
(2) $\mathfrak{p}D[X]_A$, where $\mathfrak{p} \in w\text{-}\mathrm{Max}(D[X])$ is an upper to zero in $D[X]$ disjoint from A.

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In addition, if D is integrally closed,
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Question

Is the last argument in the previous theorem true without D being integrally closed?

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Application of w-maximal spectrum of the Anderson rings with integrally closed

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Application of w-maximal spectrum of the Anderson rings with integrally closed

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Proposition (Preprint, Baek and Lim)

Let D be an integrally closed domain.
Then D is an H-domain if and only if $D[X]_A$ is an H-domain.

Consider the following two sets:

$$\begin{aligned} \mathfrak{A} &= \{MD[X]_A \mid M \in w\text{-Max}(D)\} \\ \mathfrak{B} &= \{\mathfrak{p}D[X]_A \mid \mathfrak{p} \in w\text{-Max}(D[X]) \text{ is an upper to zero in } D[X] \\ &\qquad \text{disjoint from } A\}. \end{aligned}$$

Observation of the w-maximal spectrum of Anderson rings

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- $R[X]_A$ has finite character if and only if R is a semi-quasi-local ring.

Localization of the Anderson ring at their maximal w-ideals

Let D be an integral domain.

- For any $\mathfrak{m} \in w\text{-Max}(D[X]_A)$,
  either $(D[X]_A)_{\mathfrak{m}} = D_M[X]_{N_M}$ or $(D[X]_A)_{\mathfrak{m}} = D[X]_{\mathfrak{p}}$,
  where $N_M = \{f \in D_M[X] \mid c(f) = D_M$ and
  $\mathfrak{p}$ is an upper to zero in $D[X]$ disjoint from A.

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Let D be an integral domain.

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- D is a *w-almost Dedekind domain* if
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Proposition (Preprint, Baek and Lim)

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Example
D is a w-locally Noetherian domain if and only if
$D[X]_A$ is a w-locally Noetherian domain.

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- Can we characterize the maximal w-ideals of $D[X]_A$? Yes!!
- Is there a w-local property of $D[X]_A$? w-almost Dedekind domain, w-locally Noetherian domain, ...

Thank you for your attention!!

If you have any questions or comments,
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- htbaek@knu.ac.kr