Irreducible representation of cover of Lie algebras

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Definition (Lie algebra, cf.[3])

A Lie algebra L over an arbitrary field $\mathbb{F}$ is a vector space over $\mathbb{F}$ endowed with an operation called Lie bracket satisfying the following properties :

1. Bilinearity : For $x, y, z \in L, a, b \in \mathbb{F}$
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[x, ay + bz] = a[x, y] + b[x, z]$$
2. $[x, x] = 0$ for all $x \in L$
3. Jacobi identity :
$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Example

Let V be a finite dimensional vector space over the field $\mathbb{F}$. Then V is Lie algebra with Lie bracket

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Example

$End(V)$, the set of all linear transformations on a finite dimensional vector space V over a field $\mathbb{F}$ is a Lie algebra with Lie bracket

$$[x, y] = xy - yx \text{ for } x, y \in End(V).$$

Central extensions of Lie algebras

Definition

Let $\mathcal{L}$ be a Lie algebra and $\mathcal{M}$ be an abelian Lie algebra. Then $(\mathcal{E}; f, g)$ is an extension of $\mathcal{L}$ by $\mathcal{M}$ if there exists a Lie algebra $\mathcal{E}$ such that the following is a short exact sequence :

$$0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0.$$

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$$0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0.$$

An extension $(\mathcal{E}; f, g)$ is *central* if $f(\mathcal{M}) \subseteq Z(\mathcal{E})$ where $Z(\mathcal{E})$ is the center of $\mathcal{E}$.

Example:

Consider the real Lie algebras

$$\mathcal{L} = span_{\mathbb{R}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : y, z \in \mathbb{R}\} \text{ and } \mathcal{M} = span_{\mathbb{R}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in$$

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respectively. Let

$$\mathcal{E} = span_{\mathbb{R}}\{\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R}\},$$

$f : \mathcal{M} \to \mathcal{E}$ is the identity inclusion and $g : \mathcal{E} \to \mathcal{L}$ is given by

$$g(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

Then f and g are Lie algebra homomorphisms and $ker(g) = Im(f)$. Also $f(\mathcal{M}) = \mathcal{M} = Z(\mathcal{E})$, the center of $\mathcal{E}$. Thus $0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0$ is a central extension.

Cover of a Lie algebra

Definition

Let $\mathcal{L}$ be a Lie algbera. Then a pair of Lie algebras $(\mathcal{E}, \mathcal{M})$ such that

1. $L \cong \mathcal{E}/\mathcal{M}$
2. $\mathcal{M} \subseteq Z(\mathcal{E}) \cup [\mathcal{E}, \mathcal{E}]$
3. $dim(\mathcal{E})$ is maximal

is called a maximal defining pair of $\mathcal{L}$. In a maximal defining pair $(\mathcal{E}, \mathcal{M})$, $\mathcal{E}$ is called cover of $\mathcal{L}$ and $\mathcal{M}$ is called multiplier of $\mathcal{L}$.

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Lemma

([1]) Let $\mathcal{L}$ be a Lie algebra of finite dimension n and $\mathcal{E}$ be the cover of $\mathcal{L}$. Then $dim(\mathcal{E}) \leq \frac{n(n+1)}{2}$.

We can define the cover interms of central extensions using the following lemma.

Lemma

Let $\mathcal{L}$ be a Lie algebra and $\mathcal{E}$ be its cover. Then

1. $0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0$ *is a central extension for some Lie algebra* $\mathcal{M}$
2. $Ker(g) \subseteq [\mathcal{E}, \mathcal{E}]$
3. $dim(\mathcal{E})$ *is maximal.*

Example

Consider the central extension $0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0$ of $\mathcal{L}$ by $\mathcal{M}$ as in Example 5. Here

$$[\mathcal{E}, \mathcal{E}] = span_{\mathbb{R}}\{\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{R}\}$$

and thus $Ker(g) = [\mathcal{E}, \mathcal{E}]$. Also $dim(\mathcal{E}) = 3$ which is maximal due to previous lemma. Hence, $\mathcal{E}$ is a cover of $\mathcal{L}$.

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- The *degree* of the representation ϕ is the dimension of the vector space V.
- A representation ϕ of $\mathcal{L}$ on V is *reducible* if there is proper subspace of V which is invariant under ϕ, i.e, if there exists a proper subspace W of V such that $\phi(x)(w) \in W$ for all $x \in \mathcal{L}$ and $w \in W$. Otherwise, ϕ is called *irreducible.*

Cohomology of Lie algebras

- Consider $C^n(\mathcal{L}, \mathbb{C}) = \{f : \mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L} \to \mathbb{C} |\ f \text{ is } n\text{-linear and alternating}\}$.

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- Define $\delta_n : C^n(\mathcal{L},\mathbb{C}) \to C^{n+1}(\mathcal{L},\mathbb{C})$ by

$$\delta_n(f)(x_1, x_2, \cdots, x_{n+1}) = \sum_{i<j} f([x_i, x_j], x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$$

  for $f \in C^n(\mathcal{L},\mathbb{C})$.

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Then δ_n is the coboundary map.
- One can obtain a chain
$$\cdots \xrightarrow{\delta_{n-2}} C^{n-1}(\mathcal{L}, \mathbb{C}) \xrightarrow{\delta_{n-1}} C^n(\mathcal{L}, \mathbb{C}) \xrightarrow{\delta_n} C^{n+1}(\mathcal{L}, \mathbb{C}) \xrightarrow{\delta_{n+1}} \cdots$$
with

- $Z^n(\mathcal{L}, \mathbb{C}) = Ker(\delta_n) = \{f \in C^n(L, \mathbb{C}) \mid \delta_n(f) = 0\}$
- $B^n(\mathcal{L}, \mathbb{C}) = Im(\delta_{n-1})$
$$= \{f \in C^n(L, \mathbb{C}) \mid \text{there exists } g \in C^{n-1}(\mathcal{L}, \mathbb{C}) \text{ such that } \delta_{n-1}(g) = f\}$$

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Then the n-th cohomology group is given by

$$H^n(\mathcal{L}, \mathbb{C}) = \frac{Z^n(\mathcal{L}, \mathbb{C})}{B^n(\mathcal{L}, \mathbb{C})}$$

where $Z^n(\mathcal{L}, \mathbb{C})$ and $B^n(\mathcal{L}, \mathbb{C})$ are called group of n-cocycles and group of n-coboundaries respectively.

For $n = 2$,

- The set of 2-cocycles is given by

$$Z^2(\mathcal{L}, \mathbb{C}) = \{f : \mathcal{L} \times \mathcal{L} \to \mathbb{C} : f \text{ is bilinear and } f([x,y],z) + f([y,z],x) + f([z,x],y) = 0 \ \forall x,y,z \in \mathcal{L}\}$$

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- The set of 2-coboundaries are

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- The second cohomology group of $\mathcal{L}$ is given by

$$H^2(\mathcal{L}, \mathbb{C}) = \frac{Z^2(\mathcal{L},\mathbb{C})}{B^2(\mathcal{L},\mathbb{C})}$$

Projective representation

Definition

A projective representation of a Lie algebra $\mathcal{L}$ on V is a linear map $\phi : \mathcal{L} \to \mathfrak{pgl}(V)$ such that ϕ is a Lie algebra homomorphism where $\mathfrak{pgl}(V)$ is the quotient Lie algebra $\mathfrak{gl}(V)/\{kI_V : k \in \mathbb{F}\}$.

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Example

Every linear representation of $\mathcal{L}$ is a projective representation. For, consider a linear representation $\rho : \mathcal{L} \to \mathfrak{gl}(V)$ of $\mathcal{L}$ and the natural homomorphism $\pi : \mathfrak{gl}(V) \to \mathfrak{pgl}(V)$, then the composition $\pi \circ \rho : \mathcal{L} \to \mathfrak{pgl}(V)$ is a projective representation.

Proposition

Let ϕ be a projective representation of $\mathcal{L}$ on V. Then there is a linear map $\Phi : \mathcal{L} \to \mathfrak{gl}(V)$ and a bilinear map $\alpha : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ such that

$$[\Phi(x), \Phi(y)] = \alpha(x, y) I_V + \Phi([x, y]) \quad \text{for all} \quad x, y \in \mathcal{L}. \tag{1}$$

Conversely, if there is a linear map Φ and a bilinear map α satisfying (1), then $\pi \circ \Phi : \mathcal{L} \to \mathfrak{pgl}(V)$ where $\pi : \mathfrak{gl}(V) \to \mathfrak{pgl}(V)$ is the canonical homomorphism, is a projective representation of $\mathcal{L}$.

Remark

Let $\mathcal{L}$ be a Lie algebra. If there is a linear map $\Phi : \mathcal{L} \to \mathfrak{gl}(V)$ and a bilinear map $\alpha : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ satisfying the condition

$$[\Phi(x), \Phi(y)] = \alpha(x, y) I_V + \Phi([x, y]),$$

then one can obtain a projective representation of $\mathcal{L}$.

Observation

Suppose a Lie algebra $\mathcal{L}$ admits a projective representation, then the bilinear map $\alpha : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ satisfies

$$\alpha([x, y], z) + \alpha([y, z], x) + \alpha([z, x], y) = 0. \tag{2}$$

In other words, α is a bilinear map satisfying the 2-cocycle condition. That is, $\alpha \in Z^2(\mathcal{L}, \mathbb{C})$.

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Thus the projective representation also referred to as an *α-representation* on the vector space V.

Recall *the Hochschild - Serre spectral sequence of low dimensions* ([1]). If I is an ideal of the Lie algebra $\mathcal{L}$, then there is an exact sequence of Lie algebra homomorphisms

$$0 \to I \xrightarrow{f} \mathcal{L} \xrightarrow{g} \mathcal{L}/I \to 0. \tag{3}$$

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and $s : \mathcal{L}/I \to \mathcal{L}$ is a the section of g.For an $\mathcal{L}$-module A the sequence

$$Hom(\mathcal{L}/I, A) \xrightarrow{Inf_1} Hom(\mathcal{L}, A) \xrightarrow{Res} Hom(I, A) \xrightarrow{Tra} H^2(\mathcal{L}, A) \xrightarrow{Inf_2} H^2(\mathcal{L}/I, A) \tag{4}$$

is exact and is called the Hochschild - Serre spectral sequence (Theorem 3.1, cf. [1]) , where Inf_1 and Inf_2 are inflation maps, Res is the restriction map and $Tra : Hom(I, A) \to H^2(\mathcal{L}, A)$ is the Transgression map and is defined by

$$Tra(\chi) = [\chi \circ \beta]$$

for $\chi \in Hom(I, A)$, where $\beta(x, y) = [s(x), s(y)] - s([x, y])$; s is the section of g in (3).

Theorem

Let $\mathcal{E}$ be the cover of a Lie algebra $\mathcal{L}$. Suppose $\Phi : \mathcal{L} \to \mathfrak{gl}(V)$ is α-representation of $\mathcal{L}$. If $[\alpha] \in Im(Tra)$, then there is a Lie algebra homomorphism $\Gamma : \mathcal{E} \to \mathfrak{gl}(V)$ such that $\Gamma(h)$ is a scalar multiple of the identity transformation on V for any $h \in \mathcal{L}'$.

Theorem

Consider the central extensions $0 \to \mathcal{L}' \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L} \to 0$ *and* $0 \to \mathbb{F} \xrightarrow{\mu} \mathfrak{gl}(V) \xrightarrow{\pi} \mathfrak{pgl}(V) \to 0$ *of Lie algebras. Then for all pairs of Lie algebra homomorphisms* $\Gamma : \mathcal{E} \to \mathfrak{gl}(V)$ *and* $\alpha : \mathcal{L}' \to \mathbb{F}$ *such that* $\mu \circ \alpha = \Gamma \circ f$*, there is a projective representation* $\Phi : \mathcal{L} \to \mathfrak{pgl}(V)$ *such that* $\pi \circ \Gamma = \Phi \circ g$*.*

The projective representation Φ of $\mathcal{L}$ on the space V is called *irreducible* if 0 and V are the only Φ-invariant subspaces of V.

Now let $Irr(\mathcal{E})$ denotes the set of all irreducible linear representations of $\mathcal{E}$ and $Irr^{\alpha}(\mathcal{L})$ denotes the set of all irreducible α-representations of $\mathcal{L}$ where $[\alpha] \in H^2(\mathcal{L}, \mathbb{C})$. The subsequent theorem establishes the correspondence between these sets.

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Theorem

Let $\mathcal{E}$ be the cover of a Lie algebra $\mathcal{L}$. Then there is a bijection between the sets $Irr(\mathcal{E})$ and $\bigcup_{[\alpha]\in H^2(\mathcal{L},\mathbb{C})} Irr^{\alpha}(\mathcal{L})$.

References

- P. G. Batten, “Multilpliers and covers of Lie algebras”, Ph.D Thesis, 1993.
- Arjeh M. Cohen and D. E. Taylor, “On a Certain Lie algebra Defined by a Finite Group”, The American Mathematical Monthly, Aug. - Sep., 2007.
- James E. Humphreys, “Introduction to Lie Algebras and Representation Theory”, Springer - Verlag, 1972.
- Peyman Niroomand and Francesco G. Russo, ”A Note on the Schur Multiplier of a Nilpotent Lie Algebra ”, Communications in Algebra, Volume 39, 2011 - Issue 4.
- Charles A. Weibel, An Introduction to homological algebra, Cambridge University Press, 1994.

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