Galois coverings and Krull-Gabriel dimension of algebras

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Plan of the talk.

1. Basic definitions and some results on KG dimension.
2. Krull-Gabriel dimension of repetitive categories.
3. Krull-Gabriel dimension of cluster repetitive category.
4. Krull-Gabriel dimension of weighted surface algebras.

Assume $K = \overline{K}$.

- R is a **locally bounded K-category**, that is, R is isomorphic with a bound quiver K-category of some locally finite quiver.
- $\mathrm{MOD}(R)$ is the category of **right R-modules**, that is, K-linear contravariant functors $M\colon R \to \mathrm{MOD}(K)$.
- $\mathrm{mod}(R)$ is the full subcategory of **finite dimensional R-modules**, that is, $M \in \mathrm{mod}(R)$ if $\dim M = \sum_{x \in \mathrm{ob}(R)} \dim_K M(x) < \infty$.
- $\mathcal{F}(R)$ is the category of **finitely presented** contravariant K-linear functors $T\colon \mathrm{mod}(R) \to \mathrm{mod}(K)$, that is, all T such that there is an exact sequence of functors

  $$_R(-, M) \xrightarrow{_R(-,f)} {_R(-, N)} \to T \to 0,$$

  for some $M, N, f : M \to N \in \mathrm{mod}(R)$. Thus $T \cong \mathrm{Coker}_R(-, f)$.
- $\mathcal{F}(R)$ is abelian.

Assume $\mathcal{C}$ is a essentially small abelian category with Krull-Gabriel filtration $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ indexed by ordinal numbers.

- The **Krull-Gabriel dimension** $\mathsf{KG}(\mathcal{C})$ of $\mathcal{C}$ is the smallest ordinal number α such that $\mathcal{C}_\alpha = \mathcal{C}$, if it exists. Otherwise, we set $\mathsf{KG}(\mathcal{C}) = \infty$ and say that the Krull-Gabriel dimension of $\mathcal{C}$ is **undefined**.
- If $\mathsf{KG}(\mathcal{C}) = \alpha \in \mathbb{N}$, then the Krull-Gabriel dimension of $\mathcal{C}$ is **finite**.
- We set $\mathsf{KG}(R) := \mathsf{KG}(\mathcal{F}(R))$.

General fact:

Assume $\mathcal{C}, \mathcal{D}$ are abelian categories and $F : \mathcal{C} \to \mathcal{D}$ is an exact functor.

(1) If F is full and dense, then $\mathsf{KG}(\mathcal{D}) \leq \mathsf{KG}(\mathcal{C})$.

(2) If F is faithful, then $\mathsf{KG}(\mathcal{C}) \leq \mathsf{KG}(\mathcal{D})$.

The conjecture of Prest.

The algebra A is of domestic representation type if and only if $\mathrm{KG}(A)$ is finite.

All known results support the conjecture of Prest. In particular:

- A is of finite representation type if and only if $\mathrm{KG}(A) = 0$ (Auslander '82).
- We have $\mathrm{KG}(A) \neq 1$ (Krause '98).
- If A is hereditary of Euclidean type, then $\mathrm{KG}(A) = 2$ (Geigle '86).
- A is a cycle-finite algebra of infinite representation type: A is domestic if and only if $\mathrm{KG}(A) = 2$ (Skowroński'16).
- $\mathrm{KG}(A) = \infty$ for wild (Prest '88), tubular (Geigle '86), string of non-domestic type (Schröer '00), pg-critical (Kasjan, Pastuszak '14).

Let R, A be locally bounded K-categories, G a group of K-linear automorphisms of R acting freely on $\mathrm{ob}(R)$.
Then a K-linear functor $F : R \to A$ is a **Galois covering**, if F induces an isomorphism $A \cong R/G$ where R/G is the **orbit category**.

Remarks:

- G acts on $\mathrm{mod}(R)$ as ${}^{g}M := M \circ g^{-1}$ and on homomorphisms in a natural way.
- A finite convex subcategory $\mathcal{D}_R$ of R is called the **fundamental domain** of R, if for any $M \in \mathrm{ind}(R)$ there is $g \in G$ with $\mathrm{supp}({}^{g}M) \subseteq \mathcal{D}_R$.
- The category R is **locally support-finite** (lsf), if for any object x of R the union of supports $\mathrm{supp}(N)$, for $N \in \mathrm{ind}(R)$ with $N(x) \neq 0$, is finite.
- The category R is **intervally-finite**, if any finite subcategory of R has finite convex hull.

Lemma:

(a) If there is a fundamental domain $\mathcal{D}_R$ of R, then R is lsf.

(b) If R is lsf and intervally-finite, then there exists $\mathcal{D}_R$ of R.

Theorem A. (Pastuszak '19)

Assume R is a lsf category, which is intervally-finite, G is a torsion-free admissible group of K-linear automorphisms of R and $F : R \to A$ the associated Galois covering. Then

$$\mathrm{KG}(R) = \mathrm{KG}(\mathcal{D}_R) = \mathrm{KG}(A).$$

Theorem B. (Pastuszak '23)

Assume $F : R \to A$ is a Galois covering. Then $\mathrm{KG}(R) \leq \mathrm{KG}(A)$.

Theorem (Assem-Skowroński '88,'93).

(a) The repetitive category $\widehat{A}$ of algebra A is lsf and tame if and only if $\widehat{A} \cong \widehat{B}$ where B is tilted algebra of Dynkin or Euclidean type, or tubular algebra.

(b) $\widehat{B}$ is cycle-finite. $\quad (M_0 \xrightarrow{f_1} M_1 \to \dots \xrightarrow{f_r} M_r = M_0 \Rightarrow f_1, ..., f_r \notin \mathrm{rad}_R^\infty)$

Remark: $\widehat{B}$ admits fundamental domain $\mathcal{D}_{\widehat{B}} = \begin{bmatrix} B_0 & 0 \\ D(B) & B_1 \end{bmatrix}$.

Corollary 1. (Pastuszak'19)

Let A be an algebra such that $\widehat{A}$ is lsf. Then $\mathrm{KG}(\widehat{A}) \in \{0, 2, \infty\}$ and:

(a) $\mathrm{KG}(\widehat{A}) = 0$ if and only if $\widehat{A} \cong \widehat{B}$ for B tilted of Dynkin type;

(b) $\mathrm{KG}(\widehat{A}) = 2$ if and only if $\widehat{A} \cong \widehat{B}$ for B tilted of Euclidean type;

(c) $\mathrm{KG}(\widehat{A}) = \infty$ if and only if $\widehat{A}$ is wild or $\widehat{A} \cong \widehat{B}$ for B tubular.

Ad.(b) B- Euclidean type $\Rightarrow \widehat{B}$ - cycle-finite $\Rightarrow \mathcal{D}_{\widehat{B}}$ is cycle-finite of domestic type $\Rightarrow \mathrm{KG}(\mathcal{D}_{\widehat{B}}) = 2$ (Skowroński) $\Rightarrow \mathrm{KG}(\widehat{B}) = 2$ (Thm. A for $F : \widehat{B} \to T(B)$)

Theorem (Skowroński'89).

Let A be a standard selfinjective algebra A of infinite representation type, $\widehat{B}$ is a repetitive category of an algebra B and G an infinite cyclic admissible group of K-linear automorphisms of $\widehat{B}$.

(a) A is domestic if and only if $A \cong \widehat{B}/G$, where B is tilted algebra of Euclidean type,

(b) A is non-domestic of polynomial growth if and only if $A \cong \widehat{B}/G$, where B is a tubular algebra.

Corollary 2. (Pastuszak'19)

Let A be a standard selfinjective algebra of infinite type.

(a) If A domestic, then $\mathrm{KG}(A) = 2$;

(b) If A nondomestic of polynomial growth, then $\mathrm{KG}(A) = \infty$.

Proof: Theorem A for $F : \widehat{B} \to A \cong \widehat{B}/G$ and Corollary 1

C - tilted algebra, $E = \mathrm{Ext}^2_C(DC, C)$ - C-C-bimodule

$$\check{C} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & C_{-1} & & & \\ & E_0 & C_0 & & \\ & & E_1 & C_1 & \\ 0 & & & \ddots & \ddots \end{bmatrix} \text{ – cluster repetitive category of } C$$

Identity maps $C_i \to C_{i-1}$, $E_i \to E_{i-1}$ induce an automorphism $\nu : \check{C} \to \check{C}$ and we have a Galois covering $F : \check{C} \to \check{C}/\langle\nu\rangle = \tilde{C}$

Remarks:

(a) $\tilde{C} \cong C \ltimes \mathrm{Ext}^2_C(DC, C)$ relation extension algebra

(b) Assem-Brüstle-Schiffler:

$\tilde{C}$ for tilted alg. of type Q = cluster tilted alg. $\mathrm{End}_{\mathcal{C}_Q}(T)$ of type Q

Question:

Relations between $\mathrm{KG}(\widetilde{C})$, $\mathrm{KG}(\check{C})$ and $\mathrm{KG}(\widehat{C})$?

Observations:

(1) An algebra $\mathcal{D}_{\check{C}} = \left[\begin{smallmatrix} C_0 & 0 \\ E & C_1 \end{smallmatrix}\right]$ is a fundamental domain of $\check{C} \Rightarrow \check{C}$ is lsf $\Rightarrow$ Theorem A for $F : \check{C} \to \check{C}/\langle\nu\rangle = \tilde{C} \Rightarrow \mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C})$.

(2) Theorem (Assem-Brüstle-Schiffler'08): There is an additive K-lin. fun.

$$\phi : \mathrm{mod}(\widehat{C}) \to \mathrm{mod}(\check{C})$$

which is full, dense (and exact) such that $\mathrm{Ker}(\phi)$ equals the class of all homomorphisms in $\mathrm{mod}(\widehat{C})$ which factorize through $\mathrm{add}(\mathcal{K}_C)$, where $\mathcal{K}_C = \{\widehat{P}_x, \tau^{1-i}\Omega^{-i}(C) \mid x \in (\widehat{C})_0, i \in \mathbb{Z}\}$.

(3) $\mathrm{add}(\mathcal{K}_C)$ is contravariantly finite class in $\mathrm{mod}(\widehat{C})$.

(4) ${}_{\check{C}}(\phi(-), Z) \in \mathcal{F}(\widehat{C})$ for any $Z \in \mathrm{mod}(\check{C})$.
Enough to show that ${}_{\check{C}}(\phi(-), \phi(N)) \in \mathcal{F}(\widehat{C})$ for any $N \in \mathrm{mod}(\widehat{C})$, since ϕ is dense.

This follows from (3).

(5) If $U \in \mathcal{F}(\check{C})$, then $U \circ \phi \in \mathcal{F}(\widehat{C})$.
It follows from (4) and the fact that $\mathcal{F}(\widehat{C})$ is abelian.

Summing up:

$\Lambda_\phi = (-) \circ \phi \colon \mathcal{F}(\check{C}) \to \mathcal{F}(\widehat{C})$ $\Rightarrow$ exact and faithful $\Rightarrow$ $\mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C})$

Hence $\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C})$.

Theorem (JP-Pastuszak'22).

$\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) = \mathrm{KG}(\widehat{C}) \in \{0, 2, \infty\}$, for any tilted algebra C, and the following assertions hold:

(a) C is tilted of Dynkin type if and only if $\mathrm{KG}(\widetilde{C}) = 0$.

(b) C is tilted of Euclidean type if and only if $\mathrm{KG}(\widetilde{C}) = 2$.

(c) C is tilted of wild type if and only if $\mathrm{KG}(\widetilde{C}) = \infty$.

Sketch of the proof:

- C tilted of Dynkin type $\Rightarrow \mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C}) = 0$, so $\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) = \mathrm{KG}(\widehat{C}) = 0$
- C of Euclidean type $\Rightarrow \mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C}) = 2$, but $\mathrm{KG}(\widetilde{C}) \neq 0, 1$
- C of wild type $\Rightarrow \widetilde{C}$ is also of wild type
- C either of Dynkin, or Euclidean or wild type - equivalences

Remark:

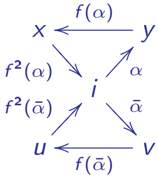
Prest conjecture is valid for cluster-tilted algebras.

A **triangulation quiver** is a pair (Q, f) such that:

(a) Q is 2-regular with involution $(\bar{\ })$ on arrows

(b) f is a permutation on arrows, $t(\alpha) = s(f(\alpha))$

(c) $f^3 = \mathrm{id}$



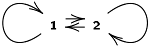
Let T be triangulation of a surface S, $\vec{T}$ an orientation of triangles. With a triangulated surface $(S, \vec{T})$ we associate a triangulation quiver $(Q(S, \vec{T}), f)$.

A **weighted surface algebra** $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ is algebra of the form KQ/I, where $(Q, f) = (Q(S, \vec{T}), f)$ is a triangulation quiver, generators of I depend on permutation f; $m_\bullet$ and $c_\bullet$ are weight and parameter functions on (Q, f), respectively.

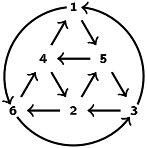
Remark:

Q does not need to be the Gabriel quiver of Λ.

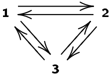
Exceptional families: disc algebras $D(\lambda)$, $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$, tetrahedral algebras $\Lambda(\lambda)$, triangle algebras $T(\lambda)$, spherical algebras $S(\lambda)$ for any $\lambda \in K^*$.



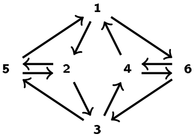
disc



tetrahedral



triangle



spherical

Theorem (Erdmann-Skowroński'20).

For Λ not isomorphic to $D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$, $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$, there exists a quotient algebra Γ of Λ which is a string algebra of non-polynomial growth.

Theorem (JP-Pastuszak'23).

Weighted surface algebras Λ have $\mathrm{KG}(\Lambda) = \infty$.

Sketch of the proof:

- $\Lambda \ncong D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$, $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$:
  Thm. ES $\Rightarrow$ faithful, exact functor $\Phi : \mathrm{mod}\Gamma \to \mathrm{mod}\Lambda \Rightarrow$
  $\mathrm{KG}(\Gamma) \leq \mathrm{KG}(\Lambda)$;
  Γ - string algebra of non-domestic type $\Rightarrow$ $\mathrm{KG}(\Gamma) = \infty$ (Schröer) $\Rightarrow$
  $\mathrm{KG}(\Lambda) = \infty$.
- $\Lambda(\lambda) \cong T(B(\lambda))$ for $B(\lambda)$ tubular algebra of type $(2,2,2,2)$ for $\lambda \neq 1$ and $B(1)$ is pg-critical

  $S(\lambda) \cong T(C(\lambda))$ for $C(\lambda)$ tubular algebra of type $(2,2,2,2)$ for $\lambda \neq 1$ and $C(1)$ is pg-critical

  $T(\lambda) \cong S(\lambda)/\mathbb{Z}_2$.

  $D(\lambda) \cong \Lambda(\lambda)/\mathbb{Z}_3$,

  By applying Theorem B in this case also $\mathrm{KG}(\Lambda) = \infty$.
- $\Lambda \cong D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$: similar arguments

A **hybrid algebra** is a block of idempotent algebra $e\Lambda e$ of weighted surface algebra Λ.

Remarks:

- The class of hybrid algebras contains all weighted surface algebras, all BGA and many other symmetric algebras of tame or finite representation type.
- The quiver (Q, f) of a hybrid algebra does not have to satisfy $f^3 = \mathrm{id}$.

Theorem.

If a quiver (Q, f) of a hybrid algebra H is a triangulation quiver, different from quiver of exceptional families of algebras, then $KG(H) = \infty$.